

# A family of SCFTs hosting all “very attractive” relatives of the $(2)^4$ Gepner model

Katrin Wendland<sup>\*†</sup>

## Abstract

This work gives a manual for constructing superconformal field theories associated to a family of smooth  $K3$  surfaces. A direct method is not known, but a combination of orbifold techniques with a non-classical duality turns out to yield such models. A four parameter family of superconformal field theories associated to certain quartic  $K3$  surfaces in  $\mathbb{CP}^3$  is obtained, four of whose complex structure parameters give the parameters within superconformal field theory. Standard orbifold techniques are used to construct these models, so on the level of superconformal field theory they are already well understood.

All “very attractive”  $K3$  surfaces belong to the family of quartics underlying these theories, that is all quartic hypersurfaces in  $\mathbb{CP}^3$  with maximal Picard number whose defining polynomial is given by the sum of two polynomials in two variables. A particular member of the family is the  $(2)^4$  Gepner model, such that these theories can be viewed as complex structure deformations of  $(2)^4$  in its geometric interpretation on the Fermat quartic.

## Introduction

Conformal field theory (CFT) should provide a natural link between mathematics and physics: While mathematical definitions of CFT are available [1, 2, 3, 4, 5, 6, 7], CFT has applications both in statistical mechanics [8, 9] and in string theory [10, 11, 12, 13]. Superconformal field theory (SCFT) can be viewed as a generalization of CFT which in many ways is better behaved, not least since all consistent string theories yield superconformally invariant field theories [14, 15]. Nevertheless, the mutual interactions between the mathematics and the physics of SCFT remain limited. On the one hand, those approaches which are fully accepted in mathematics [16, 17, 5, 7] do not succeed to embody all examples that are of importance in string theory. On the other hand, string theory has not yet matured to the status of a consistent theory in the mathematical sense of the word.

Geometric methods yield a promising vehicle to bridge this gap: In physics such methods have a good tradition, and they are built into string theory by construction. In mathematics, an area which enjoys great impact from SCFT is algebraic geometry, where e.g. mirror symmetry tells its own well-known success story [18, 19, 20]. To use geometric methods in SCFT a precise understanding of the mechanisms by which SCFT enriches geometry is desirable. The present work aims to make a contribution in that direction.

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<sup>\*</sup>Dept. of Mathematics, UNC Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599-3250, USA.

<sup>†</sup>Mathematics Institute, Univ. of Warwick, Coventry CV4 7AL, UK; wendland@maths.warwick.ac.uk.

There are only very few examples where the encoding of geometry in SCFT is understood to a satisfactory degree. When restricting to SCFTs associated to Calabi-Yau  $d$ -folds, as seems natural from a string theorist’s point of view\*, then complex tori and their orbifolds almost exhaust the list of such examples, to which the less conservative geometer will wish to add lattice and WZW models along with their orbifolds and coset models. At complex dimension  $d \geq 3$  this still means that even the degree of our ignorance is hard to gauge. On the other hand, at complex dimension  $d = 1$  with solely the elliptic curve to account for in the zoo of Calabi-Yau  $d$ -folds, the complete picture is understood.

The complex two-dimensional case resides at the borderline when accounting for ignorance: There are only two topological types of Calabi-Yau 2-folds, the complex two-torus and the  $K3$  surface. Both the moduli spaces of SCFTs associated to complex two-tori and to  $K3$  surfaces are known to a high degree of plausibility [21, 22, 23]. For complex two-tori all associated SCFTs can be constructed explicitly, and their location within the moduli space along with the translation from geometric to SCFT data is well understood [24, 21]. Within the 80-dimensional moduli space of SCFTs associated to  $K3$  surfaces, only a finite number of subvarieties of maximal dimension 16 is known in the sense that the corresponding SCFTs can be constructed explicitly (by orbifold techniques [25, 26, 27] or as Gepner models [28, 29]), and their location within the moduli space along with a translation between geometric and SCFT data is available [30, 31]. No direct method is known for the construction of SCFTs associated to smooth  $K3$  surfaces. In this work I provide such a construction for a real four parameter family of SCFTs associated to smooth quartic  $K3$  surfaces. I combine orbifold techniques with non-classical dualities thus not giving a new construction of SCFTs but rather singling out a family of theories which now is well under control from both a superconformal field theorist’s and an algebraic geometer’s point of view: The relevant theories are easy to construct as orbifolds and at the same time have a parametrization in terms of algebraic equations describing the underlying quartic  $K3$  surfaces. In fact, the latter geometric interpretation yields all four real parameters as complex structure deformations, while the complexified Kähler structure remains constant at a natural value.

The tools used here combine a detailed understanding of the moduli space of SCFTs following Aspinwall and Morrison [23] with orbifold techniques taken from the physics literature and following Dixon, Harvey, Vafa, Witten [25, 26] and Eguchi, Ooguri, Taormina, Yang [27] as well as their mathematical predecessors, where particularly in the context of  $K3$  surfaces Nikulin’s work [32] is of importance. These joint forces have already led to the appropriate description of all orbifold SCFTs obtained from toroidal models within the moduli space [30, 31], which the present work is built on. As a final ingredient Witten’s results on the phase structure of the parameter space of supersymmetric gauge theories [33] comes to aid. The family of SCFTs studied in this work allows a description within each of these settings. SCFTs associated to Calabi-Yau 2-folds are comparatively tractable because they enjoy extended  $N = (4, 4)$  supersymmetry beyond the usual  $N = (2, 2)$  supersymmetry required for SCFTs associated to Calabi-Yau  $d$ -folds in general. Geometrically this corresponds to the observation that all Calabi-Yau 2-folds are hyperkähler. Hence many of my techniques will not generalize to higher dimensions. However, the main result as stated addresses geometric interpretations of SCFTs on  $K3$  surfaces that are equipped with a complex structure, a Kähler class, and a B-field. I call these data a “refined geometric interpretation” to distinguish them from the ordinary geometric interpretations of such theories which amount to fixing a hyperkähler structure, a volume, and a B-field. Additionally specifying a complex

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\*Here and in the following I restrict my attention to unitary SCFTs in two dimensions.

structure within the data of such a theory\* amounts to the choice of an  $N = (2, 2)$  subalgebra within the given  $N = (4, 4)$  superconformal algebra (although vice versa not every choice of an  $N = (2, 2)$  superconformal algebra induces the choice of a complex structure, as we shall see and as was already pointed out in [34]). Viewed as  $N = (2, 2)$  SCFTs the main protagonist of this work, a four-parameter family of SCFTs associated to a smooth family of quartic  $K3$  surfaces, is understood to a degree which should allow for applications that may very well generalize to higher dimensions.

Particularly because this family of SCFTs simultaneously allows a description in terms of representation theory through its orbifold construction and in terms of algebraic geometry in a way which is compatible with linear sigma model constructions, it yields a tailor made testing ground for modern techniques in SCFT which so far have only been successfully applied within one of these pictures or in simpler examples like toroidal SCFTs or minimal models. Indeed, my main protagonist family of SCFTs can be viewed as a complex structure deformation of the  $(2)^4$  Gepner model in its geometric interpretation on the Fermat quartic. As such it should lend itself to a study of D-branes combining orbifold techniques as in [35] with modern techniques from matrix factorization [36, 37, 38, 39, 40, 41, 42, 43, 44], not only for  $(2)^4$  but for the entire four-parameter family of SCFTs which deforms  $(2)^4$ . In terms of more abstract approaches to SCFT it may also be interesting to study this family from the viewpoint of the chiral de Rham complex [45, 46, 47]: All relevant vertex algebras should be accessible explicitly.

Concerning the title of this work let me briefly comment on “very attractive”  $K3$  surfaces. Following Moore [48, 49], I call a  $K3$  surface attractive iff it has maximal Picard number. If an attractive  $K3$  surface can be given as zero locus of a homogeneous polynomial of degree 4 in  $\mathbb{CP}^3$  which decomposes into a sum of two polynomials in two variables each, then I call it “very attractive”. All “very attractive”  $K3$  surfaces belong to the family of quartics which I associate SCFTs to in this work, forming a dense subset. However, not all of the theories associated to “very attractive”  $K3$  surfaces are rational. This work originally arose from ideas concerning attractiveness in geometry and rationality in SCFT. In particular, part of the results presented here were already announced in [50], where however I did not notice that the constructions sketched there for “very attractive” quartics extend to a smooth family of SCFTs. There, also only three of now four real parameters  $\alpha, \beta, \beta', \gamma$  were explored. To reduce to the situation of [50], set  $\beta' = \beta$  in the present work. Finally, details and proofs were omitted in [50] which I now provide in full generality. In fact, the present work aims to be essentially self-contained.

It is organized as follows.

As a warmup, Section 1 is devoted to SCFTs associated to Calabi-Yau 1-folds. The material is well-established but is presented in a slightly more abstract form than is common, to facilitate later reference in the higher dimensional case. I give a representation theoretic definition of these theories and summarize their properties as SCFTs and in relation to the geometry and the algebraic description of elliptic curves.

Section 2 also begins with the presentation of known material concerning the moduli space of SCFTs associated to Calabi-Yau 2-folds. Again I give a representation theoretic definition of such theories, and I summarize the current state of knowledge concerning their moduli space and its relation to geometric data. Particularly the notion of refined geometric interpretations is discussed and compared to the generalized  $K3$  structures of [51, 34]. Moreover, two families of SCFTs are introduced, one associated to real four-tori and one to  $K3$  surfaces, yielding the

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\*Given a hyperkähler structure, for each compatible choice of complex structure up to normalization there exists a unique compatible Kähler class.

main protagonists of this work. Both as a preparation for the main result and as an example for the general techniques discussed before, two distinct refined geometric interpretations are worked out for each of these families.

Section 3 is devoted to the formulation of the main result of this work and its discussion: The family of SCFTs associated to  $K3$  introduced previously allows a refined geometric interpretation which associates it to a family of smooth quartic  $K3$  surfaces, given in terms of explicit algebraic equations. I provide a first step in the proof of this claim and motivate it in terms of an extension of a construction by Inose [52] to SCFTs: Inose's results concern complex structures of  $K3$  surfaces only, while on the level of SCFTs we deal with pairs of complex structures and complexified Kähler structures. Motivated by this interpretation of the main result on a purely geometric level I deduce properties of the natural Kähler class of our quartic  $K3$  surfaces in  $\mathbb{CP}^3$ , which descends from the class of the Fubini-Study metric on  $\mathbb{CP}^3$ : The induced Kähler class on a  $\mathbb{Z}_2$ -orbifold of such quartics is closely and explicitly related to a Kähler class which is induced by a Kummer construction. This result makes the underlying Kähler-Einstein metrics directly accessible to numerical approaches developed recently [53] and may be interesting in its own right. I present a simple proof which does not use results from SCFT.

The following Section 4 contains the remaining steps in the proof of the main result. This largely amounts to understanding the particular model  $(2)^4$  within the family of SCFTs discussed here, along with its deformations.

I conclude with a discussion in Section 5, and four Appendices contain details about the geometry of elliptic curves, about minimal models, and about Gepner models, which are used in the main text.

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## 1 Warmup: SCFTs associated to Calabi-Yau 1-folds

Before turning to the main topic of this work, I discuss SCFTs associated to Calabi-Yau 1-folds. There is only one topological type of Calabi-Yau 1-folds, namely the elliptic curve, and SCFTs associated to elliptic curves are well understood: These theories allow an abstract mathematical definition in terms of representation theory, all such theories can be constructed explicitly, and their moduli space is known, including the translation from conformal field theoretic into geometric data. The moduli space and the notion of geometric interpretation for such theories bear some resemblance to the corresponding notions for Calabi-Yau 2-folds, which play center stage in this work. For this reason and also since SCFTs associated to elliptic curves are building blocks of the main protagonists of this work it is worthwhile to describe these theories in some detail, even though the material is standard and can be found in various textbooks, see also [54].

Section 1.1 is devoted to the mathematical definition of SCFTs on elliptic curves and the description of these theories. While my definition is not completely standard, the expert will notice that it yields precisely those theories known as toroidal SCFTs with central charge  $c = 3 = \bar{c}$  in the physics literature. My definition has the advantage that it can be

completely paralleled when it comes to defining SCFTs associated to Calabi-Yau 2-folds. Since some background knowledge in SCFT is assumed in this section, the non-expert may choose to skip directly to Section 1.2 and accept the claims made there as given facts. That section is devoted to the discussion of the moduli space of SCFTs associated to elliptic curves, including the notion of mirror symmetry and its cousins. In Section 1.3 an algebraic description for elliptic curves is introduced which is needed later and which differs from the standard Weierstraß form.

## 1.1 Definition and properties

I use the following definition for SCFTs associated to elliptic curves:

### Definition 1.1

An  $N = (2, 2)$  SCFT  $\mathcal{E}$  with\* central charges  $c = \bar{c} = 3$  is called TOROIDAL or ASSOCIATED TO AN ELLIPTIC CURVE iff the following holds:

The pre-Hilbert space  $\mathcal{H}$  of  $\mathcal{E}$  decomposes into  $\mathcal{H} = \text{NS} \oplus \text{R}$ , where the Neveu-Schwarz sector NS and the Ramond sector R are isomorphic under the spectral flow. Moreover, in NS all charges with respect to the  $\mathfrak{u}(1)$  currents  $J, \bar{J}$  of the superconformal algebras on the left and right are integral, where the standard normalization

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2} + \text{O}(1), \quad \bar{J}(\bar{z})\bar{J}(\bar{w}) \sim \frac{\bar{c}/3}{(\bar{z}-\bar{w})^2} + \text{O}(1). \quad (1.1)$$

is used.

This definition implies that every SCFT  $\mathcal{E}$  associated to an elliptic curve contains the operators of two-fold spectral flow as fermionic fields  $\psi_{\pm}$  on the left and  $\bar{\psi}_{\mp}$  on the right, and that both the holomorphic and the anti-holomorphic W-algebras contain a  $\mathfrak{u}(1)^3$  current algebra: On each side there is one  $\mathfrak{u}(1)$  current  $J, \bar{J}$  belonging to the superconformal algebra, and two further purely bosonic currents, the superpartners  $j_{\pm}, \bar{j}_{\pm}$  of the  $\psi_{\pm}, \bar{\psi}_{\pm}$ . It is not hard to see that  $\psi_{\pm}, \bar{\psi}_{\pm}$  give an ordinary Dirac fermion, and<sup>†</sup>  $J = i:\psi_{+}\psi_{-}:.$  Moreover,  $\mathcal{E}$  is a tensor product of a bosonic toroidal CFT at  $c = \bar{c} = 2$  (most conveniently defined as CFT with central charges  $c = \bar{c} = 2$  such that both holomorphic and anti-holomorphic W-algebras contain a  $\mathfrak{u}(1)^2$  current algebra) with the fermionic theory at  $c = \bar{c} = 1$  given by the Dirac fermion. We denote the real and imaginary parts of  $\sqrt{2}j_{\pm}, \sqrt{2}\bar{j}_{\pm}$  by  $j_1, j_2, \bar{j}_1, \bar{j}_2$ , normalized such that

$$j_k(z)j_l(w) \sim \frac{\delta_{kl}}{(z-w)^2} + \text{O}(1), \quad \bar{j}_k(\bar{z})\bar{j}_l(\bar{w}) \sim \frac{\delta_{kl}}{(\bar{z}-\bar{w})^2} + \text{O}(1). \quad (1.2)$$

The left-handed Virasoro field of  $\mathcal{E}$  hence is

$$T = \frac{1}{2}:\dot{j}_1 j_1: + \frac{1}{2}:\dot{j}_2 j_2: + \frac{1}{2}:\partial\psi_{+}\psi_{-}: + \frac{1}{2}:\partial\psi_{-}\psi_{+}:. \quad (1.3)$$

A toroidal theory  $\mathcal{E}$  with central charges  $c = \bar{c} = 3$  is uniquely determined by its charge lattice  $\Gamma \subset \mathbb{R}^{2,2}$  with respect to  $(j_1, j_2; \bar{j}_1, \bar{j}_2)$ . Here  $\mathbb{R}^2$  carries the standard Euclidean scalar product, and

$$\text{for } (p; \bar{p}), (p'; \bar{p}') \in \mathbb{R}^{2,2} \text{ with } p, \bar{p}, p', \bar{p}' \in \mathbb{R}^2, \quad (p; \bar{p}) \cdot (p'; \bar{p}') = p \cdot p' - \bar{p} \cdot \bar{p}'.$$

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\*Without further mention SCFTs in this work always refer to unitary CFTs in two dimensions.

<sup>†</sup>Here and in the following, statements made for holomorphic (left-handed) fields hold analogously for anti-holomorphic (right-handed) fields, though I will not always mention this explicitly.

Each  $(p; \overline{p}) \in \Gamma$  labels a vertex operator of charge  $(p; \overline{p})$  with respect to  $(j_1, j_2; \overline{j}_1, \overline{j}_2)$ , which by (1.3) has conformal weights  $(\frac{p^2}{2}, \frac{\overline{p}^2}{2})$ . These vertex operators create the ground states with respect to the generic W-algebras of toroidal SCFTs, which are generated by the superconformal algebras together with the  $\mathfrak{u}(1)$  currents  $j_k, \overline{j}_k$ . The total partition function of such a theory and its Neveu-Schwarz part are given by

$$\begin{aligned} Z(\tau', z) &= \text{tr}_{\mathcal{H}} \left[ \frac{1}{2} \left( 1 + (-1)^F \right) y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - \frac{1}{8}} \overline{q}^{\overline{L}_0 - \frac{1}{8}} \right] \\ &= \sum_{(p; \overline{p}) \in \Gamma} \frac{q^{\frac{p^2}{2}} \overline{q}^{\frac{\overline{p}^2}{2}}}{|\eta(\tau')|^2} \cdot \frac{1}{2} \sum_{i=1}^4 \left| \frac{\vartheta_i(\tau', z)}{\eta(\tau')} \right|^2, \quad (-1)^F = e^{i\pi(J_0 - \overline{J}_0)}, \quad (1.4) \\ Z_{NS}(\tau', z) &= \text{tr}_{NS} \left[ y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - \frac{1}{8}} \overline{q}^{\overline{L}_0 - \frac{1}{8}} \right] = \sum_{(p; \overline{p}) \in \Gamma} \frac{q^{\frac{p^2}{2}} \overline{q}^{\frac{\overline{p}^2}{2}}}{|\eta(\tau')|^2} \cdot \left| \frac{\vartheta_3(\tau', z)}{\eta(\tau')} \right|^2, \end{aligned}$$

where  $\tau' \in \mathbb{H} = \{\zeta \in \mathbb{C} \mid \Im(\zeta) > 0\}$ ,  $z \in \mathbb{C}$ ,  $q = e^{2\pi i \tau'}$ ,  $y = e^{2\pi i z}$ ,  $\eta$  and  $\vartheta_i$  denote the Dedekind eta and the Jacobi theta functions, and  $J_0, \overline{J}_0, L_0, \overline{L}_0$  the respective zero modes of  $\mathfrak{u}(1)$  currents and the Virasoro fields in the superconformal algebra.

For toroidal theories associated to elliptic curves the charge lattice  $\Gamma$  can always be expressed in terms of two moduli  $\tau, \rho \in \mathbb{H}$  as follows:

$$\begin{aligned} \Gamma_{\tau, \rho} &= \left\{ \frac{1}{\sqrt{2}} (\lambda^* - B\lambda + \lambda; \lambda^* - B\lambda - \lambda) \mid \lambda^* = \sum_{k=1}^2 m_k \lambda_k^*, \lambda = \sum_{k=1}^2 n_k \lambda_k, n_k, m_k \in \mathbb{Z} \right\}, \\ \lambda_1 &:= \sqrt{\frac{\Im(\rho)}{\Im(\tau)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 := \sqrt{\frac{\Im(\rho)}{\Im(\tau)}} \begin{pmatrix} \Re(\tau) \\ \Im(\tau) \end{pmatrix}, \\ \lambda_1^* &:= \frac{1}{\sqrt{\Im(\rho)\Im(\tau)}} \begin{pmatrix} \Im(\tau) \\ -\Re(\tau) \end{pmatrix}, \quad \lambda_2^* := \frac{1}{\sqrt{\Im(\rho)\Im(\tau)}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B := \frac{\Re(\rho)}{\Im(\rho)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (1.5)$$

## 1.2 Moduli space and dualities

Since by the above every SCFT  $\mathcal{E}$  associated to an elliptic curve is uniquely determined by its charge lattice  $\Gamma_{\tau, \rho}$ , which in turn can be given in terms of a pair  $\tau, \rho \in \mathbb{H}$ , a parameter space of all such theories is  $\mathbb{H} \times \mathbb{H}$ . In fact, inspection of  $\Gamma_{\tau, \rho}$  in (1.5) shows (see [21])

### Proposition 1.2

*The moduli space of SCFTs associated to elliptic curves according to Definition 1.1 is*

$$\mathcal{M}^{\mathcal{E}} = \left( \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \times \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \right) / \mathbb{Z}_2^2,$$

where every pair  $\tau, \rho \in \mathbb{H}$  determines the unique such theory with charge lattice  $\Gamma_{\tau, \rho}$  given in (1.5),  $\text{PSL}_2(\mathbb{Z})$  acts by Möbius transforms on  $\mathbb{H}$ , and  $\mathbb{Z}_2^2$  is generated by  $U(\tau, \rho) := (\rho, \tau)$  and  $V(\tau, \rho) := (-\overline{\tau}, -\overline{\rho})$ .

Though  $U, V$  induce non-trivial actions on  $\Gamma_{\tau, \rho}$ , these agree with the actions induced by reparametrizations  $(j_1, j_2; \overline{j}_1, \overline{j}_2) \mapsto (j_1, j_2; -\overline{j}_1, \overline{j}_2)$  and  $(j_1, j_2; \overline{j}_1, \overline{j}_2) \mapsto (-j_1, j_2; -\overline{j}_1, \overline{j}_2)$ , yielding the associated SCFTs equivalent.

Traditionally, the parameter  $\tau \in \mathbb{H}$  of a SCFT  $\mathcal{E}$  associated to an elliptic curve with charge lattice  $\Gamma_{\tau, \rho}$  is interpreted as the PERIOD of an elliptic curve  $E_{\tau}$  fixing its complex structure,

while  $\rho \in \mathbb{H}$  determines a COMPLEXIFIED KÄHLER structure on  $E_\tau$ :  $\Im(\rho) > 0$  gives the volume of  $E_\tau$ , thereby specifying a Kähler structure because  $H^{1,1}(E_\tau, \mathbb{C}) \cap H^2(E_\tau, \mathbb{R}) = H^2(E_\tau, \mathbb{R}) \cong \mathbb{R}$  and in accord with  $\det(\lambda_1, \lambda_2) = \Im(\rho)$  from (1.5), while  $\Re(\rho)$  specifies the so-called B-field  $B = \Re(\rho) \cdot \lambda_1^* \wedge \lambda_2^* \in H^2(E_\tau, \mathbb{R}) \cong \mathbb{R}$ . This justifies the terminology in Definition 1.1, and the pair  $(\tau, \rho) \in \mathbb{H} \times \mathbb{H}$  specifying a toroidal SCFT is referred to as GEOMETRIC INTERPRETATION of the theory. Note that  $U(\tau, \rho) = (\rho, \tau)$  exchanges complex and complexified Kähler structures of a given geometric interpretation and thus yields the simplest form of MIRROR SYMMETRY, while  $V(\tau, \rho) = (-\bar{\tau}, -\bar{\rho})$  is induced by an orientation change of the “target space”  $E_\tau$ .

### 1.3 An algebraic description

Instead of characterizing an elliptic curve  $E_\tau$  by its period  $\tau \in \mathbb{H}$  it is often more desirable to work with explicit equations. The standard description gives an elliptic curve (with inflection point) in  $\mathbb{CP}^2$  in terms of its Weierstraß form

$$\text{with } a, b \in \mathbb{C} : \quad y^2 t = x^3 - 27axt^2 - 54bt^3 \quad \text{for } (x, y, t) \in \mathbb{CP}^2. \quad (1.6)$$

The non-degenerate elliptic curves, which I shall restrict to in the following, are the ones which obey  $a^3 \neq b^2$ . The period  $\tau \in \mathbb{H}$  of (1.6) is obtained by means of the  $j$ -function  $j: \mathbb{H} \rightarrow \mathbb{C}$ , the unique modular invariant biholomorphic function with  $q$ -expansion

$$j(\tau) = q^{-1} + 744 + 196884q + \dots, \quad q := e^{2\pi i \tau}.$$

For the curve (1.6) with period  $\tau \in \mathbb{H}$  one has

$$j(\tau) = \frac{1728a^3}{a^3 - b^2}.$$

Both for the function  $j$  and its inverse rapidly convergent algorithms are available, see e.g. [55, Section VI.9].

In the application below, the elliptic curves are given in weighted projective space,

$$E_f : \quad y_0^2 = f(y_1, y_2) \quad \text{in } \mathbb{CP}_{2,1,1}$$

with  $f$  a non-degenerate homogeneous polynomial of degree 4, i.e. such that no two roots of  $f$  agree. To arrive from the standard form (1.6) at such a description one maps the four two-torsion points to the four solutions  $(0, y_1, y_2) \in \mathbb{CP}_{2,1,1}$  of  $f(y_1, y_2) = 0$ . Without loss of generality  $f$  has the form

$$f(y_1, y_2) = y_1^4 + 2\kappa y_1^2 y_2^2 + y_2^4, \quad \kappa \in \mathbb{C} \text{ with } \Im(\kappa) \geq 0, |\kappa \pm 1| \leq 2, \quad (1.7)$$

where  $\kappa \sim -\bar{\kappa}$  if  $|\kappa \pm 1| = 2$ ,  $\kappa \sim -\kappa$  if  $\kappa \in \mathbb{R}$ , and  $\kappa = \pm 1$  gives a degenerate elliptic curve. See Figure 1.1 to picture the fundamental domain for  $\kappa$ , and see Appendix A for details. Altogether the maps  $\kappa = \kappa(\tau)$  and  $\tau = \tau(\kappa)$  which relate the algebraic description (1.7) of elliptic curves in  $\mathbb{CP}_{2,1,1}$  to their periods  $\tau \in \mathbb{H}$  amount to combining the  $j$ -function or its inverse with solving algebraic equations. Hence  $\kappa = \kappa(\tau)$  and  $\tau = \tau(\kappa)$  can be determined numerically. In some cases the result is known explicitly, e.g.

$$\tau = i \in \mathbb{H} \quad \longleftrightarrow \quad E_{f_0} : \quad y_0^2 = y_1^4 + y_2^4 \quad \text{in } \mathbb{CP}_{2,1,1}, \quad (1.8)$$

see also Appendix A.

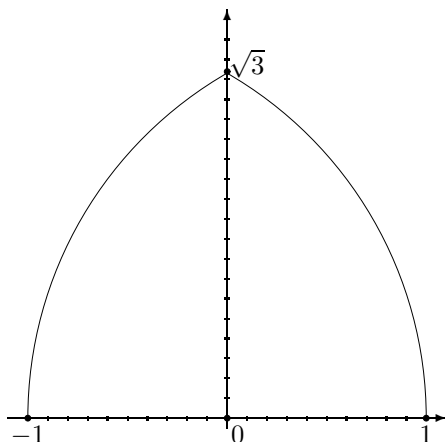


Figure 1.1: Fundamental domain for  $\kappa \in \mathbb{C}$  in  $y_1^4 + 2\kappa y_1^2 y_2^2 + y_2^4$ .

## 2 SCFTs associated to Calabi-Yau 2-folds

In this section I set the stage for the formulation of the main Result 3.1 of this work. Namely, I discuss SCFTs associated to Calabi-Yau 2-folds. These theories are defined purely within representation theory, paralleling Definition 1.1 of SCFTs associated to elliptic curves. This definition is given in Section 2.1 along with the discussion of the moduli space of SCFTs associated to Calabi-Yau 2-folds and their (refined) geometric interpretations, essentially summarizing the results of [23]. Section 2.2 is devoted to the introduction of the two families of SCFTs associated to Calabi-Yau 2-folds which feature in the main Result 3.1: One real four-parameter family of SCFTs associated to real four-tori and one associated to  $K3$  surfaces, where the latter is obtained from the former by an orbifold construction. For later convenience a pair of “dual” refined geometric interpretations for each member of both families is provided.

### 2.1 Definition and moduli space

To formally define SCFTs associated to Calabi-Yau 2-folds one can straightforwardly parallel Definition 1.1

#### Definition 2.1

An  $N = (4, 4)$  SCFT\*  $\mathcal{C}$  with central charges  $c = \bar{c} = 6$  is called ASSOCIATED TO A CALABI-YAU 2-FOLD iff the following holds:

The pre-Hilbert space  $\mathcal{H}$  of  $\mathcal{C}$  decomposes into  $\mathcal{H} = \text{NS} \oplus \text{R}$ , where the Neveu-Schwarz sector NS and the Ramond sector R are isomorphic under the spectral flow. Moreover, in NS all charges with respect to the  $\mathfrak{u}(1)$  currents of the superconformal algebras on the left and right with standard normalizations (1.1) are integral.

Every  $N = (2, 2)$  SCFT with central charges  $c = \bar{c} = 6$  which obeys the additional assumptions on the spectral flow and the  $\mathfrak{u}(1)$  charges of Definition 2.1 automatically enjoys

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\*There are various extended  $N = 4$  superconformal algebras, see [56]; the one needed here is the one which contains a single  $\mathfrak{su}(2)$  Kac-Moody algebra at level 1.



$N = (4, 4)$  supersymmetry of the type assumed in the definition, see e.g. [27]: The assumptions ensure that the operators of two-fold left and right handed spectral flows are operators of these SCFTs; one checks that at central charges  $c = \bar{c} = 6$  these operators furnish additional currents which enhance the left and the right  $\mathfrak{u}(1)$  subalgebras of the  $N = (2, 2)$  superconformal algebra to an  $\mathfrak{su}(2)$  at level 1 each, thus enhancing  $N = (2, 2)$  supersymmetry to  $N = (4, 4)$ . I nevertheless include  $N = (4, 4)$  supersymmetry in the assumptions of Definition 2.1, because these theories shall be viewed as  $N = (4, 4)$  SCFTs without an a priori choice of an  $N = (2, 2)$  subalgebra of the  $N = (4, 4)$  superconformal algebra. The condition on  $\mathfrak{u}(1)$  charges makes sense without such a choice, because all Cartan tori  $\mathfrak{u}(1) \subset \mathfrak{su}(2)$  are conjugate such that the spectrum of  $\mathfrak{u}(1)$  charges does not depend on such a choice. While definitions analogous to 1.1 and 2.1 make sense at central charges  $c = \bar{c} = 3D$  for arbitrary  $D \in \mathbb{N}$ , the cases  $D \in \{1, 2\}$  are special in that for them the moduli spaces of SCFTs associated to Calabi-Yau  $D$ -folds are known and are expected to decompose into a finite number of connected components. For  $D = 1$  we have seen this in Section 1, while for  $D = 2$  the statement is closely linked to

**Lemma 2.2**

*If  $\mathcal{C}$  is a SCFT associated to a Calabi-Yau 2-fold according to Definition 2.1, consider its*  
CONFORMAL FIELD THEORETIC ELLIPTIC GENUS

$$\mathcal{Z}(\tau', z) := \text{tr}_R \left[ (-1)^F y^{J_0} q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \right], \quad (-1)^F = e^{i\pi(J_0 - \bar{J}_0)},$$

with  $\tau' \in \mathbb{H}$ ,  $z \in \mathbb{C}$ ,  $q = e^{2\pi i \tau'}$ ,  $y = e^{2\pi i z}$ ,  $R$  denoting the Ramond sector as in Definition 2.1, and  $J_0, \bar{J}_0, L_0, \bar{L}_0$  the respective zero modes of  $\mathfrak{u}(1)$  currents and the Virasoro fields in the superconformal algebra. Then either  $\mathcal{Z}(\tau', z) \equiv 0$  or

$$\begin{aligned} \mathcal{Z}(\tau', z) &= \frac{2}{\eta^6(\tau')} (\vartheta_2^2(\tau', z) \cdot \vartheta_3^2(\tau', 0) \cdot \vartheta_4^2(\tau', 0) \\ &\quad + \vartheta_3^2(\tau', z) \cdot \vartheta_4^2(\tau', 0) \cdot \vartheta_2^2(\tau', 0) + \vartheta_4^2(\tau', z) \cdot \vartheta_2^2(\tau', 0) \cdot \vartheta_3^2(\tau', 0)), \end{aligned}$$

where  $\eta, \vartheta_i$  denote the Dedekind eta and the Jacobi theta functions. In other words, the conformal field theoretic elliptic genus  $\mathcal{Z}$  agrees with the geometric elliptic genus of a complex two-torus or a  $K3$  surface, i.e. of one of the two topologically distinct Calabi-Yau 2-folds.

A proof of this Lemma follows from the modular properties of the conformal field theoretic elliptic genus, which for SCFTs associated to Calabi-Yau 2-folds is a theta function of degree  $n = 2$  and characteristic  $(0, 0; -4\pi i n, -2\pi i \tau)$ . The proof can be found in [57] and also in [58]. Lemma 2.2 allows us to formally assign the label “torus” or “ $K3$ ” to each SCFT associated to a Calabi-Yau 2-fold by means of the conformal field theoretic elliptic genus:

**Definition 2.3**

*A SCFT associated to a Calabi-Yau 2-fold is said to be a SCFT ON A REAL FOUR-TORUS iff its conformal field theoretic elliptic genus vanishes. Otherwise, it is said to be a SCFT ON A  $K3$  SURFACE.*

While using [24, 21] one can show that the SCFTs associated to real four-tori form a connected component of the moduli space of all SCFTs associated to Calabi-Yau 2-folds, a proof of the analogous statement for SCFTs associated to  $K3$  is not known. In physics, one largely works under the assumption that indeed the space of SCFTs associated to Calabi-Yau 2-folds has only two connected components, and no counter example to this assumption is known. Below I will solely work with smooth families of such SCFTs such that the possible existence of further components of the moduli space is not relevant to the present work.

Essentially due to the extended  $N = (4, 4)$  supersymmetry of SCFTs associated to Calabi-Yau 2-folds it is possible to determine the form of each connected component of their moduli space explicitly. Here I restrict myself to stating the result; for more details see e.g. [24, 21, 22, 59, 23, 58, 30, 31]:

**Theorem 2.4** [24, 21, 22, 59, 23, 30]

*Every connected component of the moduli space of SCFTs associated to Calabi-Yau 2-folds is either of the form  $\mathcal{M}^{tori} = \mathcal{M}^0$  or  $\mathcal{M}^{K3} = \mathcal{M}^{16}$  with\**

$$\mathcal{M}^\delta \cong \mathrm{O}^+(4, 4 + \delta; \mathbb{Z}) \backslash \mathrm{O}^+(4, 4 + \delta; \mathbb{R}) / \mathrm{SO}(4) \times \mathrm{O}(4 + \delta), \quad \delta \in \{0, 16\}.$$

*There is only one connected component of the moduli space of type  $\mathcal{M}^0$  and at least one such component of type  $\mathcal{M}^{16}$ . Here,  $\mathcal{M}^{16}$  includes points where the SCFT description is expected to break down. Namely, points  $x$  in the Grassmannian  $\widetilde{\mathcal{M}}^\delta$  of positive definite oriented four-planes in  $\mathbb{R}^{4, 4+\delta}$ ,*

$$\widetilde{\mathcal{M}}^\delta = \mathrm{O}^+(4, 4 + \delta; \mathbb{R}) / \mathrm{SO}(4) \times \mathrm{O}(4 + \delta),$$

*are described by their relative position with respect to the (unique) even unimodular lattice  $\mathbb{Z}^{4, 4+\delta} \subset \mathbb{R}^{4, 4+\delta}$ . Then  $x \in \widetilde{\mathcal{M}}^{K3}$  is expected to correspond to an ill-defined SCFT iff  $x^\perp \subset \mathbb{R}^{4, 4+\delta}$  contains roots, i.e. iff there exists an  $e \in x^\perp \cap \mathbb{Z}^{4, 4+\delta}$  with  $\langle e, e \rangle = -2$ .*

Apart from the vocabulary – using Calabi-Yau 2-folds, real four-tori, and  $K3$  surfaces – the discussion, so far, has not made a connection to geometry. However, Theorem 2.4 is in accord with the expectation from string theory that every SCFT associated to a Calabi-Yau 2-fold should allow for a non-linear sigma model description on some Calabi-Yau 2-fold. Indeed,  $\mathcal{M}^{tori}$  and  $\mathcal{M}^{K3}$  agree with the moduli spaces of  $N = (4, 4)$  superconformal non-linear sigma models on real four-tori and  $K3$  surfaces, respectively, and thanks to the high amount of supersymmetry the geometry of these moduli spaces is not expected to receive quantum corrections. The key to understanding this agreement can be found in [23], and it amounts to the observation that the Grassmannians in Theorem 2.4 can be modelled on the even cohomology of the respective Calabi-Yau 2-folds:

$$H^{even}(Y, \mathbb{R}) \cong \mathbb{R}^{4, 4+\delta} \quad \text{for} \quad Y = \begin{cases} A, & \text{a real four-torus,} & \delta = 0, \\ X, & \text{a } K3 \text{ surface,} & \delta = 16. \end{cases}$$

Here and in the following  $A, X, Y$  denote the diffeomorphism types of the respective Calabi-Yau 2-folds as real four-manifolds, with all additional structure to be introduced later. Moreover, on cohomology we use the natural scalar product  $\langle \cdot, \cdot \rangle$  induced by the intersection form:

$$\forall \alpha, \beta \in H^*(Y, \mathbb{R}) : \quad \langle \alpha, \beta \rangle = \int_Y \alpha \wedge \beta.$$

With this key in hands one can interpret the identification of the spaces  $\mathcal{M}^\delta$  of Theorem 2.4 with spaces of superconformal non-linear sigma model data on Calabi-Yau 2-folds as a generalization of the following Torelli theorem for Calabi-Yau 2-folds:

**Theorem 2.5** [60, 61, 62, 63, 64]

*Complex structures on a Calabi-Yau 2-fold  $Y$  are in 1:1 correspondence with positive definite oriented two-planes  $\Omega \subset H^2(Y, \mathbb{R}) \cong \mathbb{R}^{3, 3+\delta}$  with  $\delta = 0$  for a real four-torus  $Y = A$  and*

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\*For  $\mathcal{M}^{16}$ , a mathematical proof is not known which excludes the possibility that the actual moduli space is a quotient of the one given here. However, as argued in [23], any such non-trivial quotient carries a non-Hausdorff topology, in contradiction to expectations from physics.

$\delta = 16$  for a K3 surface  $Y = X$ .

Any orthonormal basis of a positive definite oriented two-plane  $\Omega \subset H^2(Y, \mathbb{R})$  can be interpreted as giving the real and the imaginary part of a holomorphic volume form  $\tilde{\Omega} \in H^{2,0}(Y, \mathbb{C})$ , thus explaining how  $\Omega$  can encode a complex structure. In other words,  $\Omega \otimes \mathbb{C} \subset H^2(Y, \mathbb{C})$  is the orthogonal complement of the kernel of the period map. Any two-plane  $\Omega \subset H^2(Y, \mathbb{R})$  is specified by its relative position with respect to the even unimodular lattice  $H^2(Y, \mathbb{Z}) \subset H^2(Y, \mathbb{R})$ . Appropriately decomposing the four-planes featuring Theorem 2.4 into pairs of perpendicular two-planes now yields so-called refined geometric interpretations for them:

**Definition 2.6**

Given a Calabi-Yau 2-fold  $Y$ , let  $x \in H^{even}(Y, \mathbb{R})$  denote a positive definite oriented four-plane which according to Theorem 2.4 specifies a SCFT on  $Y$ . A REFINED GEOMETRIC INTERPRETATION of this SCFT is a choice of null vectors  $v^0, v \in H^{even}(Y, \mathbb{Z})$  along with a decomposition of  $x$  into two perpendicular oriented two-planes,  $x = \Omega \perp \mathfrak{U}$ , such that

$$(1) \quad \langle v^0, v^0 \rangle = \langle v, v \rangle = 0, \quad \langle v^0, v \rangle = 1, \quad \text{and} \quad (2) \quad \Omega \perp v^0, v.$$

Following [23], a refined geometric interpretation of a SCFT  $x$  on  $Y$  indeed assigns geometric data to  $x$ , in fact precisely the data needed to specify a superconformal non-linear sigma model on the complex Calabi-Yau 2-fold  $Y$ :

**Lemma/Definition 2.7**

For a Calabi-Yau 2-fold  $Y$ , let  $x \in H^{even}(Y, \mathbb{R})$  denote a positive definite oriented four-plane with refined geometric interpretation  $v^0, v \in H^{even}(Y, \mathbb{Z})$ ,  $x = \Omega \perp \mathfrak{U}$  as in Definition 2.6. Then  $v^0, v$  are naturally interpreted as generators of  $H^0(Y, \mathbb{Z})$  and  $H^4(Y, \mathbb{Z})$ , respectively, and one finds  $\omega, B \in H^{even}(Y, \mathbb{R})$  and  $V \in \mathbb{R}$  such that

$$\begin{aligned} \mathfrak{U} &= \text{span}_{\mathbb{R}} \left( \omega - \langle \omega, B \rangle v, \quad \xi_4 = v^0 + B + \left( V - \frac{1}{2} \langle B, B \rangle \right) v \right) \\ &\text{with } \omega, B \in H^2(Y, \mathbb{R}) := H^{even}(Y, \mathbb{R}) \cap (v^0)^\perp \cap (v)^\perp, \quad V \in \mathbb{R}^+, \quad \langle \omega, \omega \rangle \in \mathbb{R}^+. \end{aligned}$$

While for every refined geometric interpretation  $B$  and  $V$  are uniquely defined,  $\omega$  is unique only up to scaling. This allows to read from a refined geometric interpretation the data  $(\Omega, \omega, V, B)$  with natural interpretations in terms of a complex structure  $\Omega$  on  $Y$ , a Kähler class  $\omega$  on  $Y$  up to scaling, a volume  $V \in \mathbb{R}^+$ , and a B-field  $B \in H^2(Y, \mathbb{R})$ .

By abuse of language I also call the data  $(\Omega, \omega, V, B)$  a REFINED GEOMETRIC INTERPRETATION of a given SCFT  $x \in \widetilde{\mathcal{M}}^\delta$ .  $\mathfrak{U}$  or equivalently the data  $(\omega, V, B)$  will be referred to as COMPLEXIFIED KÄHLER STRUCTURE, and the class of  $\omega$  will be called NORMALIZED KÄHLER CLASS.

The statement of the Lemma is a consequence of the Torelli Theorem 2.5 together with a bit of linear algebra using  $\langle \omega - \langle \omega, B \rangle v, \xi_4 \rangle = 0$  and  $\langle \xi_4, \xi_4 \rangle = 2V$ . For toroidal SCFTs one checks by direct calculation that the map from non-linear sigma model data  $(\Omega, \omega, V, B)$  to  $\mathcal{M}^{tori}$  encoded in Lemma/Definition 2.7 preserves the respective natural metrics, given by the Zamolodchikov metric on  $\widetilde{\mathcal{M}}^{tori}$ . In [23] the same is claimed for  $\mathcal{M}^{K3}$ .

Recall the comments made after Definition 2.1 concerning  $N = (4, 4)$  versus  $N = (2, 2)$  supersymmetry for SCFTs associated to Calabi-Yau 2-folds  $Y$ . According to Theorem 2.4

every positive definite oriented four-plane  $x \subset H^{even}(Y, \mathbb{R})$  specifies an  $N = (4, 4)$  SCFT with central charges  $c = \bar{c} = 6$  without particular choice of an  $N = (2, 2)$  subalgebra in the  $N = (4, 4)$  superconformal algebra. There is an  $\mathbb{S}^2/\mathbb{Z}_2$  of such subalgebras both on the left and on the right, specified by the choice of an unoriented Cartan torus  $\mathfrak{u}(1) \subset \mathfrak{su}(2)$  within the  $N = 4$  superconformal algebra on each side. For later convenience, see (2.1), my conventions differ from [23, 34] in that I do not impose an orientation on  $\mathfrak{u}(1)$ , hence the division of  $\mathbb{S}^2$  by  $\mathbb{Z}_2$ . The connected components of the space of all  $N = (2, 2)$  superconformal field theories associated to Calabi-Yau 2-folds then fiber over our spaces  $\mathcal{M}^\delta$  with fibers  $\mathbb{S}^2 \times \mathbb{S}^2/\mathbb{Z}_2^2$ .

In [30, Section 1] we have given an interpretation of the four-plane  $x$  in terms of the action of the  $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{so}(4)$  subalgebra of the  $N = (4, 4)$  superconformal algebra on the space of massless fields in the respective SCFT. Since

$$\mathrm{O}^+(2, 2; \mathbb{Z}) \backslash \mathrm{O}^+(2, 2; \mathbb{R}) / \mathrm{SO}(2) \times \mathrm{O}(2) \cong (\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \times \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}) / \mathbb{Z}_2^2, \quad (2.1)$$

from this discussion one finds that the choice of an  $N = (2, 2)$  subalgebra in the  $N = (4, 4)$  superconformal algebra amounts to a choice of decomposition  $x = \Omega \perp \bar{\Omega}$  of  $x$  into two perpendicular two-planes, up to a choice of their ordering and their individual orientations. In other words, generators of  $\mathbb{Z}_2^2$  in (2.1) act by interchanging  $\Omega$  and  $\bar{\Omega}$  and by simultaneously reversing their orientations, respectively. A decomposition  $x = \Omega \perp \bar{\Omega}$  into oriented two-planes with choice of ordering amounts to a choice of an  $N = (2, 2)$  subalgebra of the  $N = (4, 4)$  superconformal algebra together with generators of its  $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$  subalgebra. In [34] the resulting ordered pairs  $(\Omega, \bar{\Omega})$  are called GENERALIZED  $K3$  STRUCTURES, and the picture from [23] drawn above is confirmed and identified with Hitchin's notion of generalized Calabi-Yau manifolds [51] in the case of Calabi-Yau 2-folds.

When working with a fixed grading  $H^{even}(Y, \mathbb{R}) = H^0(Y, \mathbb{R}) \oplus H^2(Y, \mathbb{R}) \oplus H^4(Y, \mathbb{R})$  which amounts to the choice of two null vectors  $v^0, v \in \mathbb{H}^{even}(Y, \mathbb{Z})$  as generators of  $H^0(Y, \mathbb{Z})$  and  $H^4(Y, \mathbb{Z})$  as in Definition 2.6 and subject to condition (1) in that definition, then not every ordered pair  $(\Omega, \bar{\Omega})$  of perpendicular oriented positive definite two-planes in  $H^{even}(Y, \mathbb{R})$  gives a refined geometric interpretation in terms of the data of a conformal non-linear sigma model: Condition (2) of Definition 2.6 which ensures  $\Omega \subset H^2(Y, \mathbb{R})$  is crucial to that effect. In particular, as observed in [34], one could say that not every  $N = (2, 2)$  SCFT associated to a Calabi-Yau 2-fold arises from a conformal non-linear sigma model construction on a (complex) Calabi-Yau 2-fold like this. However, if we temporarily assume that there is just one connected component of type  $\mathcal{M}^{K3}$  in the moduli space of  $N = (4, 4)$  SCFTs associated to Calabi-Yau 2-folds, then each  $N = (4, 4)$  SCFT arises from a non-linear sigma model construction: The relevant geometric data only involve the choice of a hyperkähler structure, a volume, and a B-field, not the explicit choice of a complex structure. This serves as justification for my Definition 2.1 which insists on extended  $N = (4, 4)$  supersymmetry. Huybrechts' observation amounts to the fact that in such a non-linear sigma model construction, not every choice of  $N = (2, 2)$  subalgebra can be interpreted as the choice of a complex structure on the Calabi-Yau 2-fold. However, given  $x = \Omega \perp \bar{\Omega}$ , pairs  $v^0, v \in H^{even}(Y, \mathbb{Z})$  with (1) and (2) in Definition 2.6 can always be found. In other words, as long as the grading of  $H^{even}(Y, \mathbb{R})$  is not fixed a priori, indeed every pair  $(\Omega, \bar{\Omega})$  specifying an  $N = (2, 2)$  SCFT can be interpreted in terms of non-linear sigma model data.

## 2.2 The main protagonists

Recall the definition of SCFTs associated to elliptic curves, Definition 1.1. One finds that the (fermionic) tensor product of any two such theories is a SCFT associated to a real four-torus according to Definition 2.3. Moreover, all known geometric orbifold constructions of  $K3$  surfaces from real four-tori can be extended to constructions in SCFT, producing SCFTs associated to  $K3$  as orbifolds of SCFTs associated to four-tori, see e.g. [27]. As main protagonists of the present work I introduce two families of SCFTs associated to Calabi-Yau 2-folds in this section. The first, denoted  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$ , is associated to a family of real four-tori, and each theory is obtained as a tensor product of theories associated to elliptic curves. The second, denoted  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$ , is associated to a family of  $K3$  surfaces, and each theory is obtained as an orbifold of the corresponding  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$ .

### Definition 2.8

Denote by  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  with

$$\alpha, \beta, \beta', \gamma \in \mathbb{R} \quad \text{such that} \quad \alpha, \gamma > 0 \quad \text{and} \quad \Delta := \beta^2 - 4\alpha\gamma < 0$$

the (fermionic) tensor product of the two SCFTs associated to elliptic curves with moduli  $(\tau_k, \rho_k)$  given by

$$\tau_1 = \tau_2 = i, \quad \rho_1 = \frac{-\beta + \sqrt{\Delta}}{2\alpha}, \quad \rho_2 = \frac{-\beta' + \sqrt{\Delta}}{2}.$$

By the discussion in Section 1.2 the factor theories of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  are SCFTs on elliptic curves with square fundamental cells ( $\tau_1 = \tau_2 = i$ ), with radii  $R_1, R_2$  such that  $R_1^2 = \frac{\sqrt{-\Delta}}{2\alpha}$ ,  $R_2^2 = \frac{\sqrt{-\Delta}}{2}$ , and with B-fields given by the  $-\frac{\beta}{2\alpha}$  and the  $-\frac{\beta'}{2}$ -fold of a generator of  $H^2(E_{\tau_k}, \mathbb{Z})$ , respectively. Hence  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  is a toroidal SCFT with refined geometric interpretation on a real four-torus  $A_{\alpha,\beta,\gamma}$  with the flat metric and complex and Kähler structure induced by

$$A_{\alpha,\beta,\gamma} = \mathbb{R}^4 / \Lambda_{\alpha,\beta,\gamma}, \quad \Lambda_{\alpha,\beta,\gamma} = R_1 \mathbb{Z}^2 \oplus R_2 \mathbb{Z}^2, \quad R_1^2 = \frac{\sqrt{-\Delta}}{2\alpha}, \quad R_2^2 = \frac{\sqrt{-\Delta}}{2}, \quad (2.2)$$

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4, \quad \omega_\alpha \sim dx_1 \wedge dx_2 + \alpha dx_3 \wedge dx_4$$

with respect to standard Cartesian coordinates  $x_1, \dots, x_4$  on  $\mathbb{R}^4$ . I use the standard basis  $e_1, \dots, e_4$  of  $\mathbb{R}^4$  to introduce generators  $\lambda_1 = R_1 e_1, \lambda_2 = R_1 e_2, \lambda_3 = R_2 e_3, \lambda_4 = R_2 e_4$  of  $\Lambda_{\alpha,\beta,\gamma}$  and view the vectors forming the dual basis  $\lambda_1^*, \dots, \lambda_4^*$  as generators of  $H^1(A, \mathbb{Z})$ . Hence  $H^2(A, \mathbb{Z})$  is generated by

$$v_1^0 = \lambda_1^* \wedge \lambda_3^*, \quad v_1 = \lambda_4^* \wedge \lambda_2^*, \quad v_2^0 = \lambda_2^* \wedge \lambda_3^*, \quad v_2 = \lambda_1^* \wedge \lambda_4^*, \quad v_3^0 = \lambda_1^* \wedge \lambda_2^*, \quad v_3 = \lambda_3^* \wedge \lambda_4^*, \quad (2.3)$$

where the  $v_k^0, v_l$  obey  $\langle v_k^0, v_l^0 \rangle = 0, \langle v_k, v_l \rangle = 0, \langle v_k^0, v_l \rangle = \delta_{kl}$ . For later convenience let me give the location of each  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  in the moduli space  $\mathcal{M}^{tori}$  of SCFTs associated to real four-tori, along with two refined geometric interpretations:

### Proposition 2.9

The SCFT  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  of Definition 2.8 has a refined geometric interpretation  $(\Omega_A^0, \omega_\alpha^A, V_{\alpha,\beta,\gamma}^A, B_{\alpha,\beta,\beta'}^A)$  given by

$$\Omega_A^0 = \text{span}_{\mathbb{R}}(v_1^0 + v_1, v_2^0 + v_2),$$

$$\omega_\alpha^A = v_3^0 + \alpha v_3, \quad V_{\alpha,\beta,\gamma}^A = R_1^2 R_2^2 = \gamma - \frac{\beta^2}{4\alpha}, \quad B_{\alpha,\beta,\beta'}^A = -\frac{\beta}{2\alpha} v_3^0 - \frac{\beta'}{2} v_3.$$

Within  $\mathcal{M}^{tori}$  this theory is given by the four-plane  $x_{\alpha,\beta,\beta',\gamma}^A \subset H^{even}(A, \mathbb{R})$  with

$$x_{\alpha,\beta,\beta',\gamma}^A = \text{span}_{\mathbb{R}}(\xi_1, \xi_2, \xi_3, \tilde{\xi}_4), \quad \xi_1 = v_1^0 + v_1, \quad \xi_2 = v_2^0 + v_2, \\ \xi_3 = v_3^0 + \alpha v_3 + \frac{\beta + \beta'}{2} v_4, \quad \tilde{\xi}_4 = v_4^0 + \frac{\beta - \beta'}{2} v_3 + \gamma v_4,$$

where  $v_4^0, v_4$  denote generators of  $H^0(A, \mathbb{Z})$  and  $H^4(A, \mathbb{Z})$ , respectively.

$\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  has a “mirror dual” refined geometric interpretation  $(\Omega_{\alpha,\beta,\beta',\gamma}^A, \omega_A, V_A, B_A)$  with complex structure given by the product of two elliptic curves  $E_{\rho_1} \times E_{\rho_2}$  at the moduli  $\rho_1, \rho_2$  from Definition 2.8, with normalized Kähler form  $\omega_A = -\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$  with respect to complex coordinates  $z_j$  of  $E_{\rho_j}$  such that  $-\frac{i}{2}dz_j \wedge d\bar{z}_j$  generates  $H^2(E_{\rho_j}, \mathbb{Z})$ , and with volume  $V_A = 1$  and B-field  $B_A = 0$ .

*Proof:*

That  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  has refined geometric interpretation  $(\Omega_A^0, \omega_A^A, V_{\alpha,\beta,\gamma}^A, B_{\alpha,\beta,\beta'}^A)$  follows directly from the construction of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  and the given geometric interpretation of its factor theories. From Definition 2.7 we see that  $x_{\alpha,\beta,\beta',\gamma}^A$  is generated by  $\xi_1, \xi_2$  and

$$\xi_3 = \omega_{\alpha}^A - \langle \omega_{\alpha}^A, B_{\alpha,\beta,\beta'}^A \rangle v_4, \quad \xi_4 = v_4^0 + B_{\alpha,\beta,\beta'}^A + \left( V_{\alpha,\beta,\gamma} - \frac{1}{2} \langle B_{\alpha,\beta,\beta'}^A, B_{\alpha,\beta,\beta'}^A \rangle \right) v_4.$$

One checks that  $\xi_3$  has the claimed form and  $\tilde{\xi}_4 = \xi_4 + \frac{\beta}{2\alpha} \xi_3$ .

A “mirror dual” geometric interpretation of  $x_{\alpha,\beta,\beta',\gamma}^A$  is obtained by using  $v^0 = v_2^0, v = v_2$  as generators of  $H^0(A, \mathbb{Z})$  and  $H^4(A, \mathbb{Z})$  and letting

$$\Omega_{\alpha,\beta,\beta',\gamma}^A := \text{span}_{\mathbb{R}}(\xi_3, \tilde{\xi}_4) = \text{span}_{\mathbb{R}}\left(v_3^0 + \alpha v_3 + \frac{\beta + \beta'}{2} v_4, v_4^0 + \frac{\beta - \beta'}{2} v_3 + \gamma v_4\right) \quad (2.4)$$

specify the complex structure. It immediately follows that  $V_A = 1, B_A = 0$  and  $\omega_A = v_1^0 + v_1$  in this geometric interpretation, in accord with the claim. It remains to show that  $\Omega_{\alpha,\beta,\beta',\gamma}^A$  gives the claimed complex structure. Although this should follow from consistency with the explanations given in Section 1.2, as a reality check I include the detailed argument:

I show that the orthogonal complement of the kernel of the period map in  $H^2(A, \mathbb{R})$  has generators which with respect to standard generators  $v_3^0, v_3, v_4^0, v_4$  of two copies of a hyperbolic lattice  $H$  have precisely the form given in (2.4). Because by [65, Theorem 1.14.4] the embedding  $H \oplus H \hookrightarrow H^2(A, \mathbb{Z})$  is unique up to automorphisms of  $H^2(A, \mathbb{Z})$ , (2.4) actually determines the relative position of  $\Omega_{\alpha,\beta,\beta',\gamma}^A$  with respect to  $H^2(A, \mathbb{Z})$ , such that the claim then follows from the Torelli Theorem 2.5.

Since  $\rho_k \in \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$  and  $(\tau_k, \rho_k) \sim (-\bar{\tau}_k, -\bar{\rho}_k)$  where  $\tau_k = -\bar{\tau}_k$  for  $\tau_k = i$  by Proposition 1.2, I can work with the complex two-torus  $A$  obtained as product of two elliptic curves with moduli

$$\sigma_1 = \frac{1}{\bar{\rho}_1} = \frac{-\beta + \sqrt{\Delta}}{2\gamma}, \quad \sigma_2 = -\bar{\rho}_2 = \frac{\beta' + \sqrt{\Delta}}{2}, \quad \Delta = \beta^2 - 4\alpha\gamma.$$

Similarly to [66, p. 265 ff] I have  $A = \mathbb{C}^2/L$ , where the lattice  $L \subset \mathbb{C}^2$  is generated by

$$l_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad l_3 = \begin{pmatrix} -\sigma_1 \\ 0 \end{pmatrix}, \quad l_4 = \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}.$$

With  $m^1, \dots, m^4$  the basis dual to the one given by the  $l_k$ ,

$$v_1^0 := m^1 \wedge m^3, \quad v_1 := m^4 \wedge m^2, \quad v_3^0 := m^1 \wedge m^2, \quad v_3 := m^3 \wedge m^4, \quad v_4^0 := m^2 \wedge m^3, \quad v_4 := m^1 \wedge m^4$$

generate  $H^2(A, \mathbb{Z})$ . Then the period map is given by

$$\sum_{i < j} \det(l_i l_j) m^i \wedge m^j = v_3^0 + \sigma_2 v_4 + \sigma_1 v_4^0 - \sigma_1 \sigma_2 v_3.$$

The kernel of the period map hence is generated by

$$v_1^0, \quad v_1, \quad v_3^0 - \frac{\beta - \beta'}{2} v_4 - \alpha v_3, \quad v_4^0 - \gamma v_4 - \frac{\beta + \beta'}{2} v_3,$$

and the orthogonal complement of the kernel of the period map indeed is precisely the two-plane  $\Omega_{\alpha, \beta, \beta', \gamma}^A$  of (2.4).  $\square$

By the above,  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$  has a geometric interpretation on the torus  $A_{\alpha, \beta, \gamma}$  of (2.2) which in terms of standard Cartesian coordinates  $x_1, \dots, x_4$  of  $\mathbb{R}^4$  enjoys the symmetry

$$\zeta_4: (x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, x_4, -x_3) \quad (2.5)$$

of order four. According to (2.3),  $\zeta_4$  leaves  $v_1^0 + v_1$ ,  $v_2^0 + v_2$ ,  $v_3^0$ , and  $v_3$  invariant and thus induces a map on  $H^{even}(A, \mathbb{R})$  which by the description in Proposition 2.9 leaves invariant the four-plane  $x_{\alpha, \beta, \beta', \gamma}^A \subset H^{even}(A, \mathbb{R})$  giving  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$ . This means that  $\zeta_4$  induces an automorphism of  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$ , so that a  $\mathbb{Z}_4$ -orbifold of this theory can be constructed:

**Definition 2.10**

For  $\alpha, \beta, \beta', \gamma \in \mathbb{R}$  as in Definition 2.8 let  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  denote the  $\mathbb{Z}_4$ -orbifold of the SCFT  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$ , where  $\mathbb{Z}_4$  is generated by the action induced by  $\zeta_4$ .

The theories  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  of Definition 2.10 are well understood and can be constructed explicitly without difficulty. E.g. by the results of [27] each of these theories is a SCFT associated to  $K3$  according to Definition 2.3. The results of [30, 67] allow me to describe the four-plane  $x_{\alpha, \beta, \beta', \gamma} \subset H^{even}(X, \mathbb{R})$  which specifies this theory in terms of the lattice  $H^{even}(X, \mathbb{Z})$ , using a refined geometric interpretation on the  $\mathbb{Z}_4$ -orbifold limit  $X_{\alpha, \beta, \gamma}$  of  $K3$  obtained by minimally resolving the singularities of  $A_{\alpha, \beta, \gamma}/\mathbb{Z}_4$ , where again  $\mathbb{Z}_4$  is the group generated by  $\zeta_4$ , and  $A_{\alpha, \beta, \gamma}$  carries the Kähler and complex structure induced by (2.2).

Let me first introduce some notation which I need in order to describe this refined geometric interpretation: Let  $\pi: A_{\alpha, \beta, \gamma} \rightarrow X_{\alpha, \beta, \gamma}$  denote the rational map obtained from the orbifold procedure, and  $\pi_*: H^2(A, \mathbb{R})^{\mathbb{Z}_4} \rightarrow H^2(X, \mathbb{R})$  the induced map on cohomology. Recall from [68, 30, 67] the description of the lattice  $H^2(X, \mathbb{Z})$  in terms of  $\pi_* H^2(A, \mathbb{Z})^{\mathbb{Z}_4}$  and the exceptional divisors coming from the resolution of  $A_{\alpha, \beta, \gamma}/\mathbb{Z}_4$ : Consider the action of the subgroup  $\mathbb{Z}_2$  of  $\mathbb{Z}_4$  on  $A_{\alpha, \beta, \gamma}$ . It has 16 fixed points, labeled by an affine  $\mathbb{F}_2^4$  over the field  $\mathbb{F}_2$  with two elements 0, 1, where  $i = (i_1, \dots, i_4) \in \mathbb{F}_2^4$  with  $i_k \in \{0, 1\}$  corresponds to the fixed point at  $(x_1, x_2, x_3, x_4) = \frac{1}{2} \sum_k i_k \lambda_k$ . For our  $\mathbb{Z}_4$ -orbifold we can use the same notation, where four of the fixed points listed in  $\mathbb{F}_2^4$ , namely those in  $I^{(4)} := \{(0000), (1100), (0011), (1111)\}$ , are fixed under  $\mathbb{Z}_4$ . The remaining twelve fixed points are paired to six fixed points under the  $\mathbb{Z}_2$  subgroup of  $\mathbb{Z}_4$ , where  $(i_1, i_2, i_3, i_4) \sim (i_2, i_1, i_4, i_3)$ . We denote by  $I^{(2)}$  the set of these six fixed points, i.e.  $I^{(2)} = (\mathbb{F}_2^4 - I^{(4)})/\sim$ .

From the resolution of singularities of type  $A_1$  at the fixed points with labels  $i \in I^{(2)}$  we obtain six lattice vectors  $\hat{E}_i \in H^2(X, \mathbb{Z})$ ,  $\langle \hat{E}_i, \hat{E}_i \rangle = -2$ . On the other hand, each  $i \in I^{(4)}$  gives a singularity of type  $A_3$ , yielding three lattice vectors  $\hat{E}_i^{(k)}$ ,  $k \in \{1, 2, 3\}$ , each. Their intersection matrix is the negative of the Cartan matrix of the Lie algebra  $A_3$ , while all pairwise scalar products between vectors associated to different fixed points vanish. Moreover, all the  $\hat{E}_i, \hat{E}_j^{(k)}$  are perpendicular to  $\pi_* H^2(A, \mathbb{R})^{\mathbb{Z}_4}$ .

**Proposition 2.11**

With  $\alpha, \beta, \beta', \gamma \in \mathbb{R}$  and notations as above and in particular as in Proposition 2.9 consider the  $\mathbb{Z}_4$ -orbifold SCFTs  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  of Definition 2.10.

$\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  has a refined geometric interpretation on  $X_{\alpha,\beta,\gamma} = \widetilde{A_{\alpha,\beta,\gamma}/\mathbb{Z}_4}$  given by  $(\Omega_X^0, \omega_\alpha, V_{\alpha,\beta,\gamma}, B_{\alpha,\beta,\beta'})$  as follows: With  $\hat{v}^0, \hat{v}$  generators of  $H^0(X, \mathbb{Z})$  and  $H^4(X, \mathbb{Z})$  such that  $\langle \hat{v}^0, \hat{v} \rangle = 1$ , the lattice  $\pi_* H^2(A, \mathbb{R})^{\mathbb{Z}_4} \cap H^2(X, \mathbb{Z})$  has generators  $\hat{v}_3^0 = \pi_* v_3^0$ ,  $\hat{v}_3 = \pi_* v_3$ ,  $\tilde{\Omega}_1, \tilde{\Omega}_2$  such that

$$\langle \hat{v}_3^0, \hat{v}_3^0 \rangle = \langle \hat{v}_3, \hat{v}_3 \rangle = 0, \quad \langle \hat{v}_3^0, \hat{v}_3 \rangle = 4, \quad \langle \tilde{\Omega}_1, \tilde{\Omega}_1 \rangle = \langle \tilde{\Omega}_2, \tilde{\Omega}_2 \rangle = 2, \quad \langle \tilde{\Omega}_1, \tilde{\Omega}_2 \rangle = 0,$$

and

$$\begin{aligned} \Omega_X^0 &= \text{span}_{\mathbb{R}} \left( \tilde{\Omega}_1, \tilde{\Omega}_2 \right), \quad \omega_\alpha = \hat{v}_3^0 + \alpha \hat{v}_3, \quad V_{\alpha,\beta,\gamma} = \frac{1}{4} \left( \gamma - \frac{\beta^2}{4\alpha} \right), \\ B_{\alpha,\beta,\beta'} &= -\frac{\beta}{8\alpha} \hat{v}_3^0 - \frac{\beta'}{8} \hat{v}_3 + \frac{1}{4} \check{B}_4, \quad \check{B}_4 = - \sum_{i \in I^{(2)}} \hat{E}_i - \sum_{i \in I^{(4)}} \left( \frac{3}{2} \left( \hat{E}_i^{(1)} + \hat{E}_i^{(3)} \right) + 2\hat{E}_i^{(2)} \right) \end{aligned}$$

with primitive  $\check{B}_4 \in H^2(X, \mathbb{Z})$ . The four-plane describing  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  within  $\mathcal{M}^{K3}$  is

$$x_{\alpha,\beta,\beta',\gamma} = \text{span}_{\mathbb{R}} \left( \tilde{\Omega}_1, \tilde{\Omega}_2, \hat{v}_3^0 + \alpha \hat{v}_3 + \frac{\beta + \beta'}{2} \hat{v}, 4\hat{v}^0 + \frac{\beta - \beta'}{2} \hat{v}_3 + \check{B}_4 + (\gamma + 4) \hat{v} \right).$$

$\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  has a “dual” refined geometric interpretation  $(\Omega_{\alpha,\beta,\beta',\gamma}^X, \omega_Q, V_Q, B_Q)$  with primitive  $\omega_Q \in H^2(X, \mathbb{Z})$ ,

$$\begin{aligned} \Omega_{\alpha,\beta,\beta',\gamma}^X &= \text{span}_{\mathbb{R}} \left( \hat{v}_3^0 + \alpha \hat{v}_3 + \frac{\beta + \beta'}{2} \hat{v}, 4\hat{v}^0 + \frac{\beta - \beta'}{2} \hat{v}_3 + \check{B}_4 + (\gamma + 4) \hat{v} \right), \\ \langle \omega_Q, \omega_Q \rangle &= 4, \quad V_Q = \frac{1}{2}, \quad B_Q = -\frac{1}{2} \omega_Q. \end{aligned}$$

The vectors  $\hat{v}_3^0, \hat{v}_3, \hat{v}^0, \hat{v}$  generate a primitive sublattice of  $H^2(X, \mathbb{Z})$  with quadratic form

$$\begin{pmatrix} 4h & 0 \\ 0 & h \end{pmatrix}, \quad \text{where} \quad h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

*Proof:*

The claims about the lattice  $\pi_* H^2(A, \mathbb{R})^{\mathbb{Z}_4} \cap H^2(X, \mathbb{Z})$  follow from [69], where  $\Omega_X^0$  gives the complex structure of  $X_{\alpha,\beta,\gamma}$ , see also [30, 67]. Moreover, in [30], [67, Theorem 3.3] it is proved that  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  has a geometric interpretation  $(\Omega_X^0, \pi_* \omega_\alpha^A, \frac{1}{4} V_{\alpha,\beta,\gamma}^A, \frac{1}{4} \pi_* B_{\alpha,\beta,\beta'}^A + \frac{1}{4} \check{B}_4)$  with  $\omega_\alpha^A, V_{\alpha,\beta,\gamma}^A$  and  $B_{\alpha,\beta,\beta'}^A$  as in Proposition 2.9, from which the claims about the first refined geometric interpretation are immediate. One checks  $\langle \check{B}_4, \check{B}_4 \rangle = -32$ . Using Definition 2.7 for  $x_{\alpha,\beta,\beta',\gamma}$  one thus finds generators

$$\tilde{\Omega}_1, \quad \tilde{\Omega}_2, \quad \hat{v}_3^0 + \alpha \hat{v}_3 + \frac{\beta + \beta'}{2} \hat{v}, \quad \hat{v}^0 - \frac{\beta}{8\alpha} \hat{v}_3^0 - \frac{\beta'}{8} \hat{v}_3 + \frac{1}{4} \check{B}_4 + \left( \frac{\gamma}{4} - \frac{\beta(\beta + \beta')}{16\alpha} + 1 \right) \hat{v},$$

which are seen to simplify to the form claimed.



To obtain the claimed “dual” refined geometric interpretation, for  $i \in I^{(4)}$  let  $\widehat{E}_i := \widehat{E}_i^{(1)} + 2\widehat{E}_i^{(2)} + 3\widehat{E}_i^{(3)}$  and recall from [67, Proposition 2.1] that the lattice  $H^2(X, \mathbb{Z})$  in particular contains the vectors

$$\begin{aligned} \frac{1}{2}\widetilde{\Omega}_1 &= \frac{1}{2} \left( \widehat{E}_{(0,0,0,0)} + \widehat{E}_{(1,0,0,0)} + \widehat{E}_{(0,0,0,1)} + \widehat{E}_{(0,1,0,1)} \right), \\ \frac{1}{2}\widetilde{\Omega}_2 &= \frac{1}{2} \left( \widehat{E}_{(0,0,0,0)} + \widehat{E}_{(1,0,0,0)} + \widehat{E}_{(0,0,0,1)} + \widehat{E}_{(1,0,0,1)} \right). \end{aligned}$$

Hence also

$$\begin{aligned} v_Q &:= \frac{1}{2} (\widetilde{\Omega}_1 - \widetilde{\Omega}_2) - \frac{1}{2} (\widehat{E}_{(0,1,0,1)} - \widehat{E}_{(1,0,0,1)}), \\ v_Q^0 &:= \frac{1}{2} (\widetilde{\Omega}_1 + \widetilde{\Omega}_2) + \frac{1}{2} (\widehat{E}_{(0,1,0,1)} - \widehat{E}_{(1,0,0,1)}) \end{aligned}$$

are lattice vectors, and one checks that they are null vectors obeying  $\langle v_Q^0, v_Q \rangle = 1$ . To determine the corresponding refined geometric interpretation, one first finds

$$\begin{aligned} \widetilde{\Sigma}_{\alpha,\beta,\beta',\gamma} &= x_{\alpha,\beta,\beta',\gamma} \cap (v_Q)^\perp \\ &= \text{span}_{\mathbb{R}} \left( \widetilde{\Omega}_1 + \widetilde{\Omega}_2, \widehat{v}_3^0 + \alpha \widehat{v}_3 + \frac{\beta + \beta'}{2} \widehat{v}, 4\widehat{v}^0 + \frac{\beta - \beta'}{2} \widehat{v}_3 + \check{B}_4 + (\gamma + 4) \widehat{v} \right). \end{aligned}$$

Projection onto  $H^2(X, \mathbb{R}) = H^{\text{even}}(X, \mathbb{R}) \cap (v_Q)^\perp \cap (v_Q^0)^\perp$  then shows that we can interpret

$$\Omega_{\alpha,\beta,\beta',\gamma}^X := \text{span} \left( \widehat{v}_3^0 + \alpha \widehat{v}_3 + \frac{\beta + \beta'}{2} \widehat{v}, 4\widehat{v}^0 + \frac{\beta - \beta'}{2} \widehat{v}_3 + \check{B}_4 + (\gamma + 4) \widehat{v} \right) \subset H^2(X, \mathbb{R})$$

as specifying the complex structure of this geometric interpretation, while

$$\omega_Q := 2\widetilde{\Omega}_2 + \widehat{E}_{(0,1,0,1)} - \widehat{E}_{(1,0,0,1)}$$

gives the normalized Kähler form. The latter is indeed a primitive lattice vector with  $\langle \omega_Q, \omega_Q \rangle = 4$ . Moreover,  $\widehat{v}_3^0, \widehat{v}_3, \widehat{v}^0, \widehat{v} \in H^2(X, \mathbb{Z})$ , such that the claim about the lattice that these vectors generate is immediate from the above. Next notice that  $\xi_4 := \frac{1}{2} (\widetilde{\Omega}_1 - \widetilde{\Omega}_2)$  obeys

$$x_{\alpha,\beta,\beta',\gamma} = \widetilde{\Sigma}_{\alpha,\beta,\beta',\gamma} \perp \langle \xi_4 \rangle, \quad \langle \xi_4, v_Q \rangle = 1,$$

such that in this geometric interpretation our  $K3$  surface has volume

$$V_Q = \frac{1}{2} \langle \xi_4, \xi_4 \rangle = \frac{1}{2}.$$

Finally,

$$B_Q := \xi_4 - v_Q^0 = -\frac{1}{2} (2\widetilde{\Omega}_2 + \widehat{E}_{(0,1,0,1)} - \widehat{E}_{(1,0,0,1)}) = -\frac{1}{2} \omega_Q$$

is perpendicular to both  $v_Q^0, v_Q$  and hence gives the B-field in this geometric interpretation.  $\square$

### 3 The main claim and its geometric background

I have now provided all the necessary background material to present the main result of this paper. I do so in Section 3.1: The family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  of SCFTs on  $K3$ , which is obtained by means of an orbifold construction, is given a geometric interpretation on a smooth family of smooth algebraic  $K3$  surfaces. In this geometric interpretation,  $\alpha, \beta, \beta', \gamma$  give complex structure parameters. This family of SCFTs on  $K3$  hence is well under control both from a conformal field theorist's and from an algebraic geometer's point of view. As such, it is a first example of its kind.

Section 3.1 also contains a first part of the proof of this claim. A geometric explanation arises by extending a construction due to Inose. I therefore devote Section 3.2 to a summary of Inose's work [52]. Section 3.3 explains how my main result extends Inose's construction, using a specific (crude) version of mirror symmetry. As an implication, which allows for a proof purely within geometry, I show how the natural metric on the Fermat quartic, i.e. the Kähler-Einstein metric in the class of the Fubini-Study metric on  $\mathbb{CP}^3$ , is related to an orbifold limit of a metric on a Kummer surface. This description makes the former metric accessible to numerical investigations following [53]. I therefore find it interesting in its own right and include the discussion in Section 3.3.

#### 3.1 The main result

As explained in Section 2.1, the moduli space of SCFTs associated to Calabi-Yau 2-folds is known, at least to a high degree of plausibility (the open problems were pointed out there). Section 2.2 was devoted to the discussion of two families of examples, one in each connected component of the moduli space associated to real four-tori and  $K3$  surfaces, respectively. However, further examples of such SCFTs where explicit constructions are known are severely restricted: While all SCFTs associated to real four-tori are known, along with their locations within  $\mathcal{M}^{tori}$  [24, 21], the only known constructions of SCFTs associated to  $K3$  are orbifold constructions and the Gepner construction [28, 29]. For the former, the locations within the moduli space have been worked out in [30, 67]. The latter give about 50 discrete points in the moduli space known as Gepner or Gepner type models, and for some examples the locations have been determined in [30]. However, no direct construction for SCFTs associated to smooth  $K3$  surfaces is known, let alone for a family of such surfaces. This is why I find the following result surprising:

##### Result 3.1

For  $\alpha, \beta, \beta', \gamma \in \mathbb{R}$  as in Definition 2.8, the SCFT  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  of Definition 2.10 has a refined geometric interpretation on the smooth quartic  $K3$  surface

$$X(f_1, f_2): \quad f_1(x_0, x_1) + f_2(x_2, x_3) = 0 \quad \text{in} \quad \mathbb{CP}^3,$$

where  $f_1, f_2$  are homogeneous quartic polynomials such that the elliptic curves

$$E_{f_k}: \quad y_0^2 = f_k(y_1, y_2) \quad \text{in} \quad \mathbb{CP}_{2,1,1}$$

have periods  $\rho_1, \rho_2 \in \mathbb{H}$  with

$$\rho_1 = \frac{-\beta + \sqrt{\Delta}}{2\alpha}, \quad \rho_2 = \frac{-\beta' + \sqrt{\Delta}}{2}, \quad \Delta = \beta^2 - 4\alpha\gamma$$

as in Definition 2.8, thus defining an Abelian variety

$$A(f_1, f_2) := E_{f_1} \times E_{f_2}.$$

More precisely this refined geometric interpretation carries the natural complex and Kähler structure induced by  $X(f_1, f_2) \hookrightarrow \mathbb{CP}^3$ , i.e. the normalized Kähler class is the class  $\omega_{FS}$  induced by the Fubini-Study metric on  $\mathbb{CP}^3$ , and the volume and B-field are  $V_{FS} = \frac{1}{2}$ ,  $B_{FS} = -\frac{1}{2}\omega_{FS}$ .

Section 4 is devoted to the proof of this statement. However, at this stage I can already prove the following weaker result which also gives some insight into the geometric origin of the main claim:

**Lemma 3.2**

For  $\alpha, \beta, \beta', \gamma \in \mathbb{R}$  as in Definition 2.8 consider the SCFT  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  of Definition 2.10 and its refined geometric interpretation  $(\Omega_{\alpha, \beta, \beta', \gamma}^X, \omega_Q, V_Q, B_Q)$  of Proposition 2.11. Then the complex structure  $\Omega_{\alpha, \beta, \beta', \gamma}^X$  agrees with the one of  $X(f_1, f_2) \subset \mathbb{CP}^3$  with  $X(f_1, f_2)$  as in Result 3.1. In fact this is true for any refined geometric interpretation  $(\Omega, \omega, V, B)$  of  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  with  $\Omega = \Omega_{\alpha, \beta, \beta', \gamma}^X$ .

*Proof:*

By Proposition 2.11,

$$\Omega_{\alpha, \beta, \beta', \gamma}^X = \text{span}_{\mathbb{R}} \left( \hat{v}_3^0 + \alpha \hat{v}_3 + \frac{\beta + \beta'}{2} \hat{v}, 4\hat{v}^0 + \frac{\beta - \beta'}{2} \hat{v}_3 + \check{B}_4 + (\gamma + 4) \hat{v} \right),$$

where  $\hat{v}_3^0, \hat{v}_3, \hat{v}^0, \hat{v}$  generate a primitive sublattice of  $H^2(X, \mathbb{Z})$  with signature  $(2, 2)$ , and  $\check{B}_4 \in \mathbb{H}^2(X, \mathbb{Z})$  is primitive with  $\langle \check{B}_4, \check{B}_4 \rangle = -32$ . Setting  $\hat{v}_4^0 := 4\hat{v}^0 + \check{B}_4 + 4\hat{v}$  and  $\hat{v}_4 := \hat{v}$  we obtain

$$\Omega_{\alpha, \beta, \beta', \gamma}^X = \text{span}_{\mathbb{R}} \left( \hat{v}_3^0 + \alpha \hat{v}_3 + \frac{\beta + \beta'}{2} \hat{v}_4, \hat{v}_4^0 + \frac{\beta - \beta'}{2} \hat{v}_3 + \gamma \hat{v}_4 \right), \quad (3.1)$$

where  $\hat{v}_3^0, \hat{v}_3, \hat{v}_4^0, \hat{v}_4$  generate a primitive sublattice  $\hat{\Gamma}^{2,2}$  of  $H^2(X, \mathbb{Z})$ . By the results of Proposition 2.11 this lattice is  $\hat{\Gamma}^{2,2} = \Gamma^{2,2}(4)$ , the sum of two hyperbolic lattices  $\Gamma^{2,2} = H \oplus H$  with quadratic form rescaled by a factor of 4. By [65, Theorem 1.14.4] (see also [70, Corollary 2.10]), the embedding  $\hat{\Gamma}^{2,2} \hookrightarrow H^2(X, \mathbb{Z})$  is unique up to automorphisms of  $H^2(X, \mathbb{Z})$ . Hence (3.1) fixes the location of  $\Omega_{\alpha, \beta, \beta', \gamma}^X$  within  $H^2(X, \mathbb{R})$  with respect to  $H^2(X, \mathbb{Z})$ . By the Torelli Theorem 2.5 this uniquely identifies the complex structure.

Similarly, I showed in Proposition 2.9 that the complex structure of the Abelian variety  $A(f_1, f_2) = E_{f_1} \times E_{f_2}$  is given by the two-plane  $\Omega_{\alpha, \beta, \beta', \gamma}^A \subset H^2(A, \mathbb{R})$  whose relative position with respect to  $H^2(A, \mathbb{Z})$  is specified by

$$\Omega_{\alpha, \beta, \beta', \gamma}^A = \text{span}_{\mathbb{R}} \left( v_3^0 + \alpha v_3 + \frac{\beta + \beta'}{2} v_4, v_4^0 + \frac{\beta - \beta'}{2} v_3 + \gamma v_4 \right),$$

where  $v_3^0, v_3, v_4^0, v_4$  generate a primitive sublattice  $\Gamma^{2,2} \subset H^2(X, \mathbb{Z})$  with  $\Gamma^{2,2} = H \oplus H$  as above. In [52] Inose shows that  $A(f_1, f_2)$  and  $X(f_1, f_2)$  are isogeneous, namely the Kummer surface constructed from  $A(f_1, f_2)$  is biholomorphic to a  $\mathbb{Z}_2$ -orbifold of  $X(f_1, f_2)$ . Using [52, Lemma 5.7] in conjunction with [71, Appendix §5] the complex structure of  $X(f_1, f_2)$  is thus described by a two-plane in  $H^2(X, \mathbb{R})$  which has precisely the same form as  $\Omega_{\alpha, \beta, \beta', \gamma}^A \subset H^2(A, \mathbb{R})$  but with the quadratic form of  $\Gamma^{2,2}$  rescaled by a factor of 4. Since  $\hat{\Gamma}^{2,2} = \Gamma^{2,2}(4)$ , a comparison with (3.1) completes the proof.  $\square$

The above proof is in line with the main idea of [67], where for every geometric  $G$ -orbifold construction of  $K3$  from a complex two-torus  $A$  the rational map  $\pi: A \rightarrow X = \widetilde{A/G}$

induced from the orbifold construction was studied. More precisely, the induced map  $\pi_*: H^2(A, \mathbb{R})^G \rightarrow H^2(X, \mathbb{R})$  was extended to the total even cohomology  $H^{even}(A, \mathbb{R})^G$ . The result [67, (4.1)] shows that the vectors  $\hat{v}_4^0, \hat{v}_4$  used in the proof of the above Lemma 3.2, see (3.1), are the images of the vectors  $v^0, v \in H^{even}(A, \mathbb{Z})$  under  $\pi_*$  which in the geometric interpretation  $(\Omega_A^0, \omega_\alpha^A, V_{\alpha, \beta, \gamma}^A, B_{\alpha, \beta, \beta'}^A)$  of  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$  in Proposition 2.9 on  $A_{\alpha, \beta, \gamma}$  generate  $H^0(A, \mathbb{Z})$  and  $H^4(A, \mathbb{Z})$ .

The result of Lemma 3.2 is part of the claimed Result 3.1. It seems to imply that the “dual” refined geometric interpretation of  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  in Proposition 2.11 is the desired one. Indeed, Lemma 3.2 says that the complex structure of that refined geometric interpretation is as wanted, and Proposition 2.11 confirms that its Kähler structure, volume, and B-field are in accord with the claim in Result 3.1. One would hence like to show that  $\omega_Q$  in Proposition 2.11 is the Kähler class induced by the Fubini-Study metric of  $\mathbb{CP}^3$ . However, lattice calculations alone cannot yield such a proof, and I cannot claim  $\omega_Q = \omega_{FS}$ . The necessary additional ingredients are explained in Section 4.1, and the proof is completed in Section 4.2.

The use of Inose’s work [52] gives a lead to understand the geometry underlying Result 3.1, which I shall follow on in Section 3.2. Before doing so, let me put the statement of the result into context. Namely, a main ingredient in the proof was the fact that all relevant complex structures are given by two-planes  $\Omega_{\alpha, \beta, \beta', \gamma}^Y$  ( $Y = X$  or  $Y = A$ ) which can be specified in terms of lattices of signature  $(p, q) = (2, 2)$ , and that such lattices have unique embeddings into  $H^2(Y, \mathbb{Z})$  as primitive sublattices by Nikulin’s results [65]. Here,  $p, q < 3$  is crucial; in fact, the two-planes  $\Omega_{\alpha, \beta, \beta', \gamma}^Y$  are generated by lattice vectors iff  $\alpha, \beta, \beta', \gamma \in \mathbb{Q}$ , i.e. for a dense subset of the parameter space. In other words, iff  $\alpha, \beta, \beta', \gamma \in \mathbb{Q}$ , then  $A(f_1, f_2)$  and  $X(f_1, f_2)$  are attractive according to

### Definition 3.3

*A Calabi-Yau 2-fold  $Y$  with complex structure given by a two-plane  $\Omega_Y \subset H^2(Y, \mathbb{R})$  which is generated by lattice vectors in  $H^2(Y, \mathbb{Z})$  is called ATTRACTIVE.*

*If  $X$  is an attractive K3 surface with complex structure given by  $\Omega_X \subset H^2(X, \mathbb{R})$ , and if the quadratic form of the lattice  $\Omega_X \cap H^2(X, \mathbb{Z})$  is  $4Q_A$  with  $Q_A$  an even integral quadratic form on that lattice, then  $X$  with this complex structure is called VERY ATTRACTIVE.*

For the family  $A(f_1, f_2) = E_{f_1} \times E_{f_2}$  of Abelian varieties with  $E_{f_k}$  as in Result 3.1 we see from (2.4) that for  $\alpha, \beta, \gamma \in \mathbb{Z}$  and  $\beta' = \beta$  the quadratic form of  $\Omega_{\alpha, \beta, \beta', \gamma}^A \cap H^2(A, \mathbb{Z})$  simply is  $\begin{pmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{pmatrix}$ . This form however changes dramatically as  $\alpha, \beta, \beta', \gamma$  vary in  $\mathbb{Q}$ .

In the original mathematics literature, attractive Calabi-Yau 2-folds are called singular. Since this word can be misleading, I follow Moore’s suggested terminology. In [49, 48] Moore identifies such complex structures as attractor points for the dynamical systems associated to extremal static spherically symmetric supersymmetric black holes, which explains his terminology, see also [72]. The “very attractive” terminology of Definition 3.3 is justified because Inose shows in [52, Theorem 1] that “very attractive” K3 surfaces are the special attractive K3 surfaces of the form  $X(f_1, f_2)$ . For the latter, the quadratic form of  $\Omega_{X(f_1, f_2)} \cap H^2(X, \mathbb{Z})$  is  $4Q_A$  where  $Q_A$  is the quadratic form of  $\Omega_{A(f_1, f_2)} \cap H^2(A, \mathbb{Z})$ . By the above the “very attractive” K3 surfaces are dense in the family  $X(f_1, f_2)$ . This statement makes sense even though it is known [73] that the moduli space of complex structures on K3 does not carry a Hausdorff topology: We are varying surfaces  $X(f_1, f_2)$  in  $\mathbb{CP}^3$ , giving complex structures with a fixed polarization. In other words, in effect we are varying “marked pairs” of complex and Kähler structures (c.f. [74, p. 335]), and their moduli space is indeed Hausdorff [74, Theorem VIII.12.3].

A note on rationality: A SCFT  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  of Result 3.1 is rational iff the underlying toroidal SCFT  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  is rational. For the latter, equivalently the two tensor factors giving SCFTs associated to elliptic curves are rational. A SCFT associated to an elliptic curve with geometric interpretation given by  $\tau, \rho \in \mathbb{H}$  is rational iff there exists  $D \in \mathbb{Q}$  such that  $\tau, \rho \in \mathbb{Q}(\sqrt{-D})$  (see [48]). In our example the parameters  $\tau_1 = i = \tau_2$  for the two tensor factors are fixed, so  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  and thereby  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  is rational iff  $\rho_1, \rho_2 \in \mathbb{Q}(i)$ , or equivalently  $\alpha, \beta, \beta', \sqrt{-\Delta} \in \mathbb{Q}$ . It is easy to find examples of  $\alpha, \beta, \beta', \gamma \in \mathbb{Q}$  such that  $\Delta = \beta^2 - 4\alpha\gamma$  has  $\sqrt{-\Delta} \notin \mathbb{Q}$ . In other words, by Proposition 2.11 one finds examples of *non-rational* SCFTs in  $\mathcal{M}^{K3}$  that are described by a four-plane  $x_{\alpha,\beta,\beta',\gamma} \subset H^{even}(X, \mathbb{R})$  which is generated by lattice vectors in  $H^{even}(X, \mathbb{Z})$ . This contradicts one of the many beliefs about the relation between rationality of SCFTs and the rôle of the lattice  $H^{even}(X, \mathbb{Z}) \subset H^{even}(X, \mathbb{R})$ .

### 3.2 Inose's construction

To allow insight into the geometry underlying Result 3.1, let me briefly summarize Inose's work [52]. Choose two homogeneous polynomials  $f_1, f_2$  of degree 4 in two variables each. I will assume that  $f_1, f_2$  are non-degenerate, i.e. that they do not have multiple roots. As in Result 3.1 these polynomials define elliptic curves

$$E_{f_k} : y_0^2 = f_k(y_1, y_2) \quad \text{in} \quad \mathbb{CP}_{2,1,1}$$

with moduli  $\rho_1, \rho_2 \in \mathbb{H}$ , see Section 1.3 and Appendix A.  $\rho_1, \rho_2$  can always be brought into the form used in Result 3.1. The polynomials  $f_1, f_2$  also define a smooth quartic  $K3$  surface

$$X(f_1, f_2) : f_1(x_0, x_1) + f_2(x_2, x_3) = 0 \quad \text{in} \quad \mathbb{CP}^3.$$

Note that all the surfaces  $X(f_1, f_2)$  share the symplectic automorphism  $\sigma$  given by

$$\sigma : (x_0, x_1, x_2, x_3) \mapsto (-x_0, -x_1, x_2, x_3). \quad (3.2)$$

This automorphism generates a group  $\langle \sigma \rangle$  of order 2. Now let  $Y(f_1, f_2)$  denote the  $K3$  surface obtained by blowing up the eight nodal singularities of  $X(f_1, f_2)/\langle \sigma \rangle$ . On the other hand let  $\text{Km}(E_{f_1} \times E_{f_2})$  denote the  $K3$  surface obtained from the Abelian variety  $A(f_1, f_2) = E_{f_1} \times E_{f_2}$  by the Kummer construction. In other words, we represent  $E_{f_1} \times E_{f_2}$  as  $\mathbb{C}^2/\sim$  with standard coordinates  $(z_1, z_2)$  and  $z_k \sim z_k + 1 \sim z_k + \rho_k$ , to obtain a natural  $\mathbb{Z}_2$  action by multiplication by  $-1$  on  $\mathbb{C}^2$ . Now  $\text{Km}(E_{f_1} \times E_{f_2})$  is obtained by blowing up the sixteen nodal singularities of  $E_{f_1} \times E_{f_2}/\mathbb{Z}_2$ . Hiroshi Inose has discovered

**Theorem 3.4** [52, Theorem 2]

*The  $K3$  surface  $Y(f_1, f_2)$  obtained from  $X(f_1, f_2)/\langle \sigma \rangle$  by minimally resolving all singularities is canonically biholomorphic to the Kummer surface  $\text{Km}(A(f_1, f_2))$  of the Abelian variety  $A(f_1, f_2) = E_{f_1} \times E_{f_2}$ .*

The geometric situation found in [52] is as follows: Denote the roots of  $f_l(1, \zeta) = 0$  by  $\zeta_j^l \in \mathbb{C}$  where for later convenience I use indices  $j \in \mathbb{F}_2^2 = \{00, 10, 01, 11\}$ . The quartic  $X(f_1, f_2)$  contains sixteen lines

$$\tilde{E}_{jk} : \{ (x_0, x_1, x_2, x_3) \mid x_1 = \zeta_j^1 x_0, x_3 = \zeta_k^2 x_2 \}.$$

The four lines  $\tilde{E}_{j00}, \dots, \tilde{E}_{j11}$  intersect in the fixed point  $\tilde{F}_j = (1, \zeta_j^1, 0, 0)$  of  $\sigma$ , while the four lines  $\tilde{E}_{00k}, \dots, \tilde{E}_{11k}$  intersect in the fixed point  $\tilde{G}_k = (0, 0, 1, \zeta_k^2)$  of  $\sigma$ , forming a constellation as depicted in Figure 3.1. Though the figure is not entirely suggestive, the  $\tilde{F}_j, \tilde{G}_k$  are the

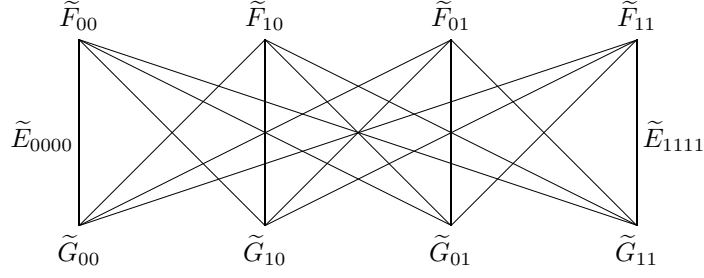


Figure 3.1: The sixteen lines  $\tilde{E}_{jk} = \tilde{F}_j \tilde{G}_k$  in  $X(f_1, f_2)$ .

only intersection points of any two lines  $\tilde{E}_{lm}$ . These lines are mapped onto themselves under the linear automorphism  $\sigma$ . In the resolved orbifold  $Y(f_1, f_2)$  they therefore give rational curves  $E_{jk}$ , while the fixed points  $\tilde{F}_j, \tilde{G}_k$  of  $\sigma$  are blown up giving eight exceptional rational curves  $F_j, G_k$ . Altogether Inose finds a double Kummer pencil in  $Y(f_1, f_2)$  as shown in Figure 3.2. Moreover, the sixteen rational curves  $E_{jk}$  can be identified with the sixteen

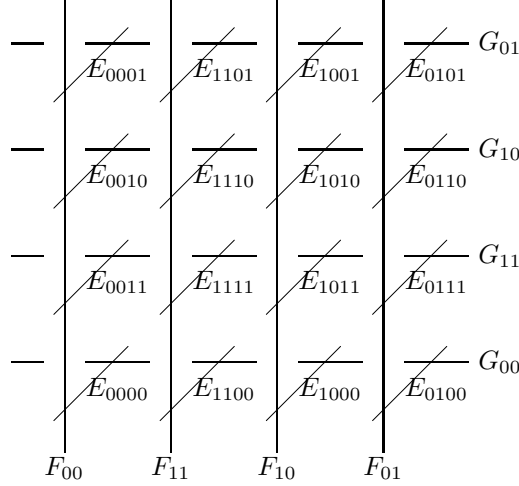


Figure 3.2: The double Kummer pencil in  $Y(f_1, f_2)$ .

irreducible components of the exceptional divisor in  $\text{Km}(E_{f_1} \times E_{f_2})$ , while the  $F_j, G_k$  can be interpreted in terms of two-cycles of  $E_{f_1} \times E_{f_2} = \mathbb{C}^2 / \sim$ . Namely,

$$\begin{aligned} \text{in } H_2(X, \mathbb{Z}) : \quad \forall j, k \in \mathbb{F}_2^2 : \quad & 2F_j + E_{j00} + E_{j10} + E_{j01} + E_{j11} = M_{12}, \\ & 2G_k + E_{00k} + E_{10k} + E_{01k} + E_{11k} = M_{34}, \end{aligned} \quad (3.3)$$

where  $M_{12}$  is the class of the image of the cycle  $z_1 = \text{const.}$  in  $E_{f_1} \times E_{f_2} = \mathbb{C}^2 / \sim$ , while the class  $M_{34}$  gives the image of the cycle  $z_2 = \text{const.}$

### 3.3 Inose’s construction extended?

In view of Inose’s geometric insights I can reformulate the Result 3.1 as follows: Since by Proposition 2.9 each  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  has a refined geometric interpretation on the Abelian variety  $A(f_1, f_2)$ , and  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  is claimed to have a refined geometric interpretation on  $X(f_1, f_2)$ , the assertion amounts to the SCFTs  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$ ,  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$ , and the ordinary  $\mathbb{Z}_2$ -orbifold  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}/\mathbb{Z}_2$  to provide an extension of Inose’s construction to the realm of Calabi-Yau 2-folds with complexified Kähler structures, in other words to the realm of SCFTs. That such extensions should exist is in itself not surprising. However, surprisingly both on  $X(f_1, f_2)$  and on  $A(f_1, f_2)$  the most natural Kähler structures turn out to occur, the ones arising from  $X(f_1, f_2) \hookrightarrow \mathbb{CP}^3$  and  $A(f_1, f_2) \cong \mathbb{C}^2/\sim$ , respectively. In contrast, note that every “very attractive” quartic (see Definition 3.3) is biholomorphic to some Kummer surface; for instance  $\mathcal{C}_{1,0,0,1}$  has a geometric interpretation on the (very attractive) Fermat quartic

$$X_{Fermat} = X(f_0, f_0): \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \quad \text{in} \quad \mathbb{CP}^3 \quad (3.4)$$

by Result 3.1 and (1.8) but nevertheless cannot be constructed from any toroidal model by a  $\mathbb{Z}_2$ -orbifold procedure [30, p. 123]. Furthermore, it is not obvious that in any given extension of Inose’s picture all associated SCFTs can be constructed explicitly, let alone by geometric orbifolds like the one yielding  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  from  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$ . To understand why this is possible in the present case, note that assuming Result 3.1 it follows that  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}/\mathbb{Z}_2$  is also a  $\mathbb{Z}_2$ -orbifold of  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$ . For every CFT  $\mathcal{C}$ , an orbifold  $\mathcal{C}/G$  by a solvable group  $G$  enjoys an action of  $G$  such that orbifolding  $\mathcal{C}/G$  by  $G$  reproduces the original CFT  $\mathcal{C}$  [75, p. 126]. It follows that any extension of Inose’s construction to the level of SCFTs must yield theories associated to  $X(f_1, f_2)$  which are obtained from theories associated to  $A(f_1, f_2)$  by an orbifold by a group of order 4, i.e. by a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  or by a  $\mathbb{Z}_4$ -action on a family of toroidal SCFTs. Choosing the complexified Kähler structure  $\omega_A = -\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ ,  $V_A = 1, B_A = 0$  of Proposition 2.9 on  $A(f_1, f_2)$  ensures that the associated SCFTs  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  all enjoy an automorphism of order 4, namely the  $\mathbb{Z}_4$ -symmetry which is induced by the geometric  $\mathbb{Z}_4$ -action  $\zeta_4$  of (2.5) in the “mirror dual” refined geometric interpretation of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  on  $A_{\alpha,\beta,\gamma}$  as in (2.2) and with B-field  $B_{\alpha,\beta,\beta'}$  as in Proposition 2.8.

In geometry, the reversal of an orbifold construction by another orbifold is of course impossible. Indeed, if  $\mathcal{C}$  is a CFT associated to some Calabi-Yau variety  $Y$  and  $G$  is a solvable symmetry group of  $\mathcal{C}$  which is induced by symplectic automorphisms of  $Y$ , then  $\mathcal{C}/G$  is a SCFT associated to  $\widetilde{Y/G}$  [25, 26], and the symmetry of type  $G$  which  $\mathcal{C}/G$  enjoys and which yields  $\mathcal{C}$  back under orbifolding is not a geometric symmetry of  $\widetilde{Y/G}$ . Such symmetries are known as QUANTUM SYMMETRIES.

That the theories  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  in Result 3.1 are nevertheless obtained by a geometric orbifold construction from the theories  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  is a consequence of the geometric re-interpretation of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  described in Proposition 2.9. Indeed, one of the crucial ideas from the early days of mirror symmetry is the observation that mirror symmetry is a non-classical equivalence between SCFTs, which interchanges the rôles of geometric and quantum symmetries [20]. With an appropriate version of mirror symmetry it should therefore be possible to find a geometric interpretation of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}/\mathbb{Z}_2$  which has a geometric  $\mathbb{Z}_2$ -symmetry that upon orbifolding yields  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$ . Because the ordinary  $\mathbb{Z}_2$ -orbifold of a toroidal SCFT  $\mathcal{T}$  descends to the Kummer construction in every refined geometric interpretation of  $\mathcal{T}$ , we can equivalently expect to find a refined geometric interpretation of  $\mathcal{T}_{\alpha,\beta,\beta',\gamma}$  such that the total symmetry group of order 4 which upon orbifolding yields  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  acts geometrically. Result 3.1 consequently claims that this desired refined geometric interpretation is the “mirror dual” of the

one on  $A(f_1, f_2)$  with complexified Kähler structure  $(\omega_A, V_A, B_A)$ , as given in Proposition 2.9, and that this geometric action is the standard action (2.5) of  $\mathbb{Z}_4$ .

It is indeed natural to view the two geometric interpretations  $(\Omega_A^0, \omega_\alpha^A, V_{\alpha, \beta, \gamma}^A, B_{\alpha, \beta, \gamma}^A)$  and  $(\Omega_{\alpha, \beta, \beta', \gamma}^A, \omega_A, V_A, B_A)$  of Proposition 2.9 as mirror duals: On the one hand, by the proof of Proposition 2.9 exchanging these two geometric interpretations amounts to interchanging the modular parameters  $\tau_k, \rho_k$  that specify the two tensor factor theories of  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$  which are SCFTs associated to elliptic curves, where according to Section 1.2 a version of mirror symmetry is given by  $U(\tau_k, \rho_k) = (\rho_k, \tau_k)$ . On the other hand, according to Proposition 2.9 the exchange of the two geometric interpretations of  $\mathcal{T}_{\alpha, \beta, \beta', \gamma}$  amounts to interchanging the rôle of the two-planes  $\Omega_A^0$  and  $\Omega_{\alpha, \beta, \beta', \gamma}^A$  which  $x_{\alpha, \beta, \beta', \gamma}^A$  decomposes into, i.e. indeed to interchanging complex and complexified Kähler structures. Note furthermore that the  $\mathbb{Z}_4$ -orbifold procedure yielding the K3 surface  $X_{\alpha, \beta, \gamma} = \widetilde{A_{\alpha, \beta, \gamma}}/\mathbb{Z}_4$  from  $A_{\alpha, \beta, \gamma} = \mathbb{R}^4/\Lambda_{\alpha, \beta, \gamma}$  of (2.2) can indeed be performed in terms of two consecutive  $\mathbb{Z}_2$ -orbifolds, the first one of which is the Kummer construction.

I hope to have convinced the reader that Result 3.1 does have a natural interpretation as extension of Inose's construction to the realm of SCFTs. However, in the above explanation I have used a very crude version of mirror symmetry, which amounts to interchanging the two-planes of a refined geometric interpretation of a SCFT but does not address the choice of null vectors as needed within any refined geometric interpretation according to Definition 2.6. Likewise, the notion of “quantum symmetries” was used in a slightly obscure fashion without proper definition, and in particular without giving a procedure to distinguish between “geometric” and “quantum” symmetries. Hence the above can only serve as a motivation, not as a proof of Result 3.1.

On purely geometric grounds the above discussion naturally raises the question whether Inose's construction can be extended to the level of Kähler-Einstein metrics. More precisely, the class of the most natural Kähler structure on  $X(f_1, f_2)$  is the class  $\omega_{FS} \in H^2(X, \mathbb{Z})$  of the Fubini-Study metric on  $\mathbb{CP}^3$ ,

$$\omega_{FS}(x) = \frac{i}{2\pi} \partial \bar{\partial} \log \left( \sum_{j=0}^3 |x_j|^2 \right) \quad \text{for } x = (x_0, x_1, x_2, x_3) \in \mathbb{CP}^3.$$

By the Calabi-Yau theorem [76] there is a unique Kähler-Einstein metric on  $X(f_1, f_2)$  with Kähler class  $\omega_{FS}$ . Since  $\omega_{FS}$  is invariant under  $\sigma$  it descends to a class  $\widehat{\omega}_{FS}$  on  $Y(f_1, f_2)$ . The class  $\widehat{\omega}_{FS}$  in turn represents the orbifold limit of an Einstein metric on  $Y(f_1, f_2)$ , which assigns vanishing volume to all components of the exceptional divisor in the resolution of  $X(f_1, f_2)/\langle \sigma \rangle$ :

$$\forall j, k \in \mathbb{F}^2 : \quad \langle \widehat{\omega}_{FS}, \widehat{F}_j \rangle = \langle \widehat{\omega}_{FS}, \widehat{G}_k \rangle = 0 \quad (3.5)$$

with  $\widehat{F}_j, \widehat{G}_k$  denoting the Poincaré duals of  $F_j, G_k$ , respectively (see Figure 3.2). One also checks

$$\langle \widehat{F}_j, \widehat{M}_{12} \rangle = \langle \widehat{G}_k, \widehat{M}_{34} \rangle = 0, \quad \langle \widehat{F}_j, \widehat{M}_{34} \rangle = \langle \widehat{G}_k, \widehat{M}_{12} \rangle = 1, \quad \langle \widehat{F}_j, \sum_{l \in \mathbb{F}_2^4} \widehat{E}_l \rangle = \langle \widehat{G}_k, \sum_{l \in \mathbb{F}_2^4} \widehat{E}_l \rangle = 4 \quad (3.6)$$

with  $\widehat{E}_l, l \in \mathbb{F}_2^4$  denoting the Poincaré duals of the rational curves  $E_l$ , and  $\widehat{M}_j$  obtained as Poincaré duals of the classes  $M_j$  introduced in (3.3),

$$\widehat{M}_j \in H^2(X, \mathbb{Z}), \quad \langle \widehat{M}_j, \widehat{M}_j \rangle = 0, \quad \langle \widehat{M}_{12}, \widehat{M}_{34} \rangle = 2,$$



see e.g. [32]. From what was said above one expects that it should be possible to express  $\widehat{\omega}_{FS}$  in terms of the simpler geometry of the Kummer surface  $\text{Km}(E_{f_1} \times E_{f_2})$ . Indeed, the result is remarkably simple\*:

**Proposition 3.5**

Let  $\widehat{\omega}_{FS}, \widehat{\omega}_{\text{Km}}$  represent the orbifold limits of Kähler-Einstein metrics on  $X(\widehat{f_1, f_2})/\langle\sigma\rangle = \text{Km}(A(f_1, f_2))$  induced by the Kähler-Einstein metric with class  $\omega_{FS}$  of the Fubini-Study metric on  $X(f_1, f_2)$ , and  $\omega_A$ , the class of the Euclidean metric on  $\mathbb{C}^2$  with  $A(f_1, f_2) = \mathbb{C}^2/\sim$ , respectively. Then

$$\widehat{\omega}_{FS} = 2\widehat{\omega}_{\text{Km}} - \frac{1}{2} \sum_{i \in \mathbb{F}_2^4} \widehat{E}_i.$$

*Proof:*

The key to the proof is the use of the explicit identifications of cycles [52] given in Section 3.2, along with a study of symplectic automorphisms of the Fermat quartic hypersurface  $X_{\text{Fermat}} = X(f_0, f_0)$  of (3.4). Indeed, one checks

$$\widehat{\omega}_{\text{Km}} = \widehat{M}_{12} + \widehat{M}_{34},$$

and since  $\widehat{\omega}_{\text{Km}}, \widehat{\omega}_{FS} \in H^2(Y, \mathbb{Z})$  do not change while  $f_1, f_2$  vary, a proof of the claim for the  $\langle\sigma\rangle$ -orbifold of the Fermat quartic  $X_{\text{Fermat}}$  is sufficient. The group  $G_{\text{Fermat}}$  of symplectic automorphisms of  $X_{\text{Fermat}}$  is well known. It is generated by phase symmetries

$$[n_0, \dots, n_3]: (x_0, \dots, x_3) \longmapsto (i^{n_0}x_0, \dots, i^{n_3}x_3) \quad \text{with } n_k \in \mathbb{Z}/4\mathbb{Z} \text{ and } \sum_k n_k \equiv 0 \pmod{4}$$

along with permutations  $\gamma \in S_4$  of the coordinates accompanied by phase symmetries  $[n_0, \dots, n_3]$  as above such that  $\sum_k n_k \equiv (1 - \det \gamma) \pmod{4}$ . Since  $[1, \dots, 1]$  acts trivially on  $\mathbb{CP}^3$  we find  $G_{\text{Fermat}} \cong \mathbb{Z}_4^3 \rtimes S_4/\mathbb{Z}_4$ .

The commutant of  $\sigma \in G_{\text{Fermat}}$  in  $G_{\text{Fermat}}$  gives the group  $\mathbb{Z}_4^3 \rtimes D_4/\mathbb{Z}_4$  with  $\mathbb{Z}_4^3$  as before and generators  $r, s$  of  $D_4$ , which acts as the dihedral group of order 8:

$$\begin{aligned} r: (x_0, x_1, x_2, x_3) &\longmapsto (x_2, x_3, x_1, -x_0), \\ s: (x_0, x_1, x_2, x_3) &\longmapsto (x_1, -x_0, x_2, x_3). \end{aligned}$$

Each element of this group  $\mathbb{Z}_4^3 \rtimes D_4/\mathbb{Z}_4$  induces a symplectic automorphism on the orbifold  $Y_{\text{Fermat}}$  of  $X_{\text{Fermat}}$  by  $\langle\sigma\rangle$ . However,  $\mathbf{1}$  and  $\sigma \in \mathbb{Z}_4^3 \rtimes D_4/\mathbb{Z}_4$  induce the trivial automorphism, leaving us with the group  $G_{\text{Km}} = (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$  generated by  $t_{1100} := [1, 3, 0, 0]$ ,  $t_{1000} := s$ ,  $r_{12} := r \circ s$ ,  $r_{13} := [1, 0, 0, 3] \circ r^2$ , with notations as above. For later convenience note

$$\begin{aligned} \text{in } \mathbb{CP}^3/\langle\sigma\rangle: \quad t_{1100}: (x_0, x_1, x_2, x_3) &\longmapsto (ix_0, -ix_1, x_2, x_3), \\ t_{1000}: (x_0, x_1, x_2, x_3) &\longmapsto (x_1, -x_0, x_2, x_3), \\ r_{12}: (x_0, x_1, x_2, x_3) &\longmapsto (x_2, x_3, x_0, x_1), \\ r_{13}: (x_0, x_1, x_2, x_3) &\longmapsto (-ix_1, x_0, x_3, ix_2). \end{aligned}$$

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\*In [53] a family of Ricci-flat Kähler-Einstein metrics is determined numerically which by Proposition 3.5 turns out to approach the one represented by  $\widehat{\omega}_{FS}$ . In fact, the explicit form of Proposition 3.5 arose as a conjecture from a discussion with the authors Matthew Headrick and Toby Wiseman of [53], and I am grateful to them for raising the relevant questions that led to this observation.

Let us now investigate these automorphisms in the light of the interpretation of  $Y_{Fermat}$  as Kummer surface as in Theorem 3.4. More precisely, I will determine the action on the cycles  $F_j, G_k, E_l, j, k \in \mathbb{F}_2^2, l \in \mathbb{F}_2^4$ , introduced above. I will in particular be interested in those cycles which are invariant under the entire group  $G_{K_m}$ , since the class  $\omega_{FS}$  is invariant under  $G_{Fermat}$  and hence the class  $\widehat{\omega}_{FS}$  which we wish to express in terms of the Kummer geometry is Poincaré dual to a cycle which is invariant under  $G_{K_m}$ . All the rational curves  $F_j, G_k, E_l$  are uniquely determined by the positions of the fixed points  $\tilde{F}_j, \tilde{G}_k$  of  $\sigma$ . Because  $G_{K_m}$  acts projectively linearly, it suffices to determine the action of  $G_{K_m}$  on these fixed points. To this end denote by  $\varepsilon$  a primitive eighth root of unity such that  $\varepsilon^2 = i$ . Then for the Fermat quartic we denote the roots  $\zeta_j^l$  of the quartic polynomial  $f_0(1, \zeta) = 1 + \zeta^4 = 0$  by

$$\zeta_{00}^l = \varepsilon, \quad \zeta_{11}^l = -\varepsilon, \quad \zeta_{10}^l = i\varepsilon, \quad \zeta_{01}^l = -i\varepsilon.$$

Consider the action of  $t_{1100}$ . The fixed points  $\tilde{G}_k = (0, 0, 1, \zeta_k^2)$  are invariant under this automorphism, while it interchanges  $\tilde{F}_{00}$  with  $\tilde{F}_{11}$ , and  $\tilde{F}_{10}$  with  $\tilde{F}_{01}$ , respectively. In other words,  $t_{1100}$  acts by a shift by (1100) on the index set  $\mathbb{F}_2^4$  of the  $E_l$ . Similarly,  $t_{1000}$  leaves the fixed points  $\tilde{G}_k = (0, 0, 1, \zeta_k^2)$  invariant. It interchanges  $\tilde{F}_{00}$  with  $\tilde{F}_{10}$ , and  $\tilde{F}_{01}$  with  $\tilde{F}_{11}$ , respectively. In other words,  $t_{1000}$  acts by a shift by (1000) on the index set  $\mathbb{F}_2^4$  of the  $E_l$ . The action of  $r_{12}$  is most easily determined - it acts on the indices  $l \in \mathbb{F}_2^4$  of the  $E_l$  by the permutation  $(l_1, l_2, l_3, l_4) \mapsto (l_3, l_4, l_1, l_2)$ . Finally, one checks that  $r_{13}$  leaves  $\tilde{F}_{00}, \tilde{F}_{11}, \tilde{G}_{00}, \tilde{G}_{11}$  invariant while interchanging  $\tilde{F}_{10}$  with  $\tilde{F}_{01}$ , and  $\tilde{G}_{10}$  with  $\tilde{G}_{01}$ . In other words,  $r_{13}$  acts on the indices  $l \in \mathbb{F}_2^4$  of the  $E_l$  by the permutation  $(l_1, l_2, l_3, l_4) \mapsto (l_2, l_1, l_4, l_3)$ . Translating into cohomology by means of the Poincaré duality we find a two-dimensional subspace of  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$  which is invariant under all of  $G_{K_m}$ , with generators

$$\widehat{\omega}_{K_m} = \widehat{M}_{12} + \widehat{M}_{34} \quad \text{and} \quad \widehat{E} := \sum_{l \in \mathbb{F}_2^4} \widehat{E}_l, \quad \langle \widehat{\omega}_{K_m}, \widehat{E} \rangle = 0.$$

On the other hand, recall that for an algebraic K3 surface  $X$  with group  $G$  of symplectic automorphisms the dimension of the  $G$ -invariant subspace  $(H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}))^G$  of  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$  can be determined by purely combinatorial methods [77]. Namely, the group  $G$  induces an action on the total rational cohomology  $H^*(X, \mathbb{Q})$  given by a so-called Mathieu representation [77, Theorem 1.4], which implies

$$\dim_{\mathbb{Q}} H^*(X, \mathbb{Q})^G = \mu(G) \quad := \quad \frac{1}{|G|} \sum_{g \in G} \mu(\text{ord}(g)),$$

$$\text{where for } n \in \mathbb{N} : \mu(n) := \frac{24}{n \prod_{\substack{p \text{ prime,} \\ p|n}} (1 + \frac{1}{p})}$$

[77, Proposition 3.4]. Since  $G$  acts symplectically, we have

$$\dim_{\mathbb{Q}} H^*(X, \mathbb{Q})^G = \dim_{\mathbb{R}} H^*(X, \mathbb{R})^G = \dim_{\mathbb{C}} H^*(X, \mathbb{C})^G.$$

By the definition of symplectic automorphisms  $H^*(X, \mathbb{C})^G \supset H^0(X, \mathbb{C}) \oplus H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C}) \oplus H^{2,2}(X, \mathbb{C})$ , so

$$\dim_{\mathbb{R}} (H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R}))^G = \mu(G) - 4. \quad (3.7)$$

For our group  $G_{\text{Km}}$  one checks

$$\mu(G_{\text{Km}}) = \frac{1}{64} (\mu(1) + 27\mu(2) + 36\mu(4)) = \frac{24}{64} \left( 1 + \frac{27}{3} + \frac{36}{6} \right) = 6.$$

By (3.7) this implies that  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$  has only a two-dimensional  $G_{\text{Km}}$  invariant subspace which hence is generated by  $\widehat{\omega}_{\text{Km}}$  and  $\widehat{E}$ . Since the form  $\widehat{\omega}_{FS} \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$  is also invariant under  $G_{\text{Km}}$  we can make an ansatz

$$\widehat{\omega}_{FS} = \lambda (\widehat{\omega}_{\text{Km}} + \alpha \widehat{E}) = \lambda (\widehat{M}_{12} + \widehat{M}_{34} + \alpha \widehat{E}).$$

Now (3.5) and (3.6) imply  $\alpha = -\frac{1}{4}$ . Moreover, since  $\widehat{\omega}_{FS}$  is the image of a primitive lattice vector  $\omega_{FS} \in H^2(X, \mathbb{Z})$  in integral cohomology,  $\widehat{\omega}_{FS}$  is a primitive lattice vector. Since  $\frac{1}{2}\widehat{E} \in H^2(X, \mathbb{Z})$  is primitive and  $\langle \widehat{M}_j, \widehat{E} \rangle = 0$  with indecomposable  $\widehat{M}_j$  spanning a primitive sublattice of  $H^2(X, \mathbb{Z})$  [32], we find  $\lambda = \pm 2$ . Finally,  $\lambda = 2$  follows since  $\widehat{\omega}_{FS}$  is a Kähler class and all  $E_l$  are effective, thus  $\langle \widehat{\omega}_{FS}, \widehat{E}_l \rangle > 0$  for all  $l \in \mathbb{F}_4^*$ .  $\square$

The above proof shows that the action of  $G_{\text{Km}}$  on  $F_j, G_k, E_l$  could be induced by the group  $G_{\text{Kummer}}^+ \cong \mathbb{F}_2^4 \rtimes \mathbb{Z}_2^2 \cong G_{\text{Km}}$  of automorphisms that fix the orbifold singular metric of a Kummer surface constructed from  $E_{f_0} \times E_{f_0}$ , and which descend from automorphisms of this variety [30, Theorem 2.7]. Namely, see Appendix A for a proof of the well-known identification  $E_{f_0} \cong \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$  which implies  $E_{f_0} \times E_{f_0} \cong \mathbb{C}^2 / (\mathbb{Z}^2 \oplus i\mathbb{Z}^2)$ . Then elements of  $\mathbb{F}_2^4 \subset G_{\text{Kummer}}^+$  act by shifts by half periods on  $\mathbb{C}^2 / \sim$ , while  $\mathbb{Z}_2^2 \subset G_{\text{Kummer}}^+$  is generated by  $(z_1, z_2) \mapsto (z_2, -z_1)$  and  $(z_1, z_2) \mapsto (iz_1, -iz_2)$ . Although it seems likely that indeed  $G_{\text{Km}} = G_{\text{Kummer}}^+$ , note that we have only compared the actions on a sublattice of  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$  of rank 18, while for the Fermat quartic  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$  has maximal rank 20. Hence we cannot conclude that the actions of the two groups agree. Luckily the observation that the actions agree on  $F_j, G_k, E_l$  has turned out to suffice to prove Proposition 3.5.

## 4 Proof of the main result

In this section I complete the proof of the main Result 3.1 of this work. The proof consists of three steps, the first one of which I have already taken in Lemma 3.2 where I proved that for  $\alpha, \beta, \beta', \gamma \in \mathbb{R}$  as in Definition 2.8 the SCFT  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  of Definition 2.10 allows a geometric interpretation  $(\Omega_{\alpha, \beta, \beta', \gamma}^X, \omega_Q, V_Q, B_Q)$  with  $\Omega_{\alpha, \beta, \beta', \gamma}^X$  the complex structure of the quartic K3 surface  $X(f_1, f_2) \subset \mathbb{CP}^3$  specified by  $\alpha, \beta, \beta', \gamma$  as described in Result 3.1. As a second step I show in Section 4.1 that Result 3.1 holds for one special member of the family  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$ , namely for  $\mathcal{C}_{1,0,0,1}$ : This model agrees with the Gepner model  $(2)^4$ , for which indeed by a combination of results by Witten [33] and Aspinwall and Morrison [23] the claim follows. The third and final step of the proof, which I explain in Section 4.2, uses the observation that the geometric interpretation of  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  in which I have already found the desired complex structure by Lemma 3.2, has complexified Kähler structure which is independent of  $\alpha, \beta, \beta', \gamma$ . I identify the relevant deformations of  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  induced by varying  $\alpha, \beta, \beta', \gamma$  and show that they are compatible with keeping the complexified Kähler structure  $(\omega_{FS}, V_{FS}, B_{FS})$  which was found for  $(2)^4$  in Section 4.1 constant for the entire family.

## 4.1 The Gepner model $(2)^4$

This section is devoted to a detailed study of one special member of the family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  of SCFTs introduced in Definition 2.10, namely the model  $\mathcal{C}_{1,0,0,1}$  obtained from the toroidal SCFT  $\mathcal{T}_{1,0,0,1}$  on the standard torus  $A_{1,0,1} = \mathbb{R}^4/\mathbb{Z}^4$  with vanishing B-field by the  $\mathbb{Z}_4$ -orbifold procedure. As a first step, I rewrite this model in a form which is more familiar to a certain class of string theorists:

**Proposition 4.1** [30, Theorem 3.5]

*The  $\mathbb{Z}_4$ -orbifold CFT  $\mathcal{C}_{1,0,0,1}$  of Definition 2.10 agrees with the  $(2)^4$  Gepner model.*

For a brief primer on Gepner models and its building blocks, the minimal models, see Appendix B and Appendix C. Specifically the models that are relevant for Proposition 4.1 are discussed in Appendix D. Proposition 4.1 was conjectured in [27] and a proof was given in [30]. It is based on an explicit field theory calculation which in fact simplifies when one uses the identifications discussed in Appendix D: The Gepner model  $(2)^2$  agrees with the SCFT associated to the elliptic curve  $\mathbb{R}^2/\mathbb{Z}^2$  with complex structure given by introducing a complex coordinate  $z = x_1 + ix_2$ , where  $x_1, x_2$  are the standard Cartesian coordinates on  $\mathbb{R}^2$ , and with vanishing B-field. Though this identification is well-known [78], Appendix D.1 recalls the explicit field identifications. I show in Appendix D.2 how these identifications imply that the ‘‘Gepner orbifold’’ (D.3), which gives  $(2)^2 \otimes (2)^2/\mathbb{Z}_4 = (2)^4$ , is induced by the geometric  $\mathbb{Z}_4$ -action (2.5) on the four-torus  $A_{1,0,1} = \mathbb{R}^4/\mathbb{Z}^4$  which underlies the toroidal model  $(2)^2 \otimes (2)^2 = \mathcal{T}_{1,0,0,1}$ , showing  $\mathcal{T}_{1,0,0,1}/\mathbb{Z}_4 = (2)^4$ .

This construction also allows to explicitly identify some of the deformations of the model  $(2)^4$ , which will become useful below. Namely, in SCFT, integrable deformations which preserve superconformal invariance are given in terms of fields of conformal dimensions  $h = \bar{h} = \frac{1}{2}$  and with  $u(1)$ -charges  $(Q, \bar{Q})$  such that  $|Q| = |\bar{Q}| = 1$ , since the superpartners of these fields are the integrable  $(h, \bar{h}) = (1, 1)$  marginal operators [79]. For example,  $(2)^2$  possesses four such linearly independent fields, as can be seen from (B.3):

$$\psi_{\pm} \bar{\psi}_{\pm} = \Phi_{\mp 2,2;\pm 2,2}^0 \otimes \Phi_{\mp 2,2;\pm 2,2}^0, \quad \psi_{\pm} \bar{\psi}_{\mp} = \Phi_{\mp 2,2;\mp 2,2}^0 \otimes \Phi_{\mp 2,2;\mp 2,2}^0,$$

where  $\psi_{\pm}, \bar{\psi}_{\pm}$  denote the left- and the right handed Dirac fermions as in Appendix D.1, and where I have used (D.1). This is in accord with the dimension 4 of the moduli space of SCFTs associated to elliptic curves as stated in Proposition 1.2. In  $\mathcal{T}_{1,0,0,1} = (2)^2 \otimes (2)^2$ , these fields also give deformations, where only

$$\begin{aligned} V_{\pm}^{(1)} &:= \Phi_{\pm 2,2;\pm 2,2}^0 \otimes \Phi_{\pm 2,2;\pm 2,2}^0 \otimes \Phi_{0,0;0,0}^0 \otimes \Phi_{0,0;0,0}^0, \\ V_{\pm}^{(2)} &:= \Phi_{0,0;0,0}^0 \otimes \Phi_{0,0;0,0}^0 \otimes \Phi_{\pm 2,2;\pm 2,2}^0 \otimes \Phi_{\pm 2,2;\pm 2,2}^0 \end{aligned} \quad (4.1)$$

are invariant under the action (D.3) which yields the  $\mathbb{Z}_4$ -orbifold  $(2)^2 \otimes (2)^2/\mathbb{Z}_4 = (2)^4$ . In other words,  $V_{\pm}^{(1)}$  and  $V_{\pm}^{(2)}$  are the deformations that  $(2)^4$  has in common with  $(2)^2 \otimes (2)^2$ , and we have

**Proposition 4.2**

*The fields  $V_{\pm}^{(1)}, V_{\pm}^{(2)}$  of (4.1) give deformations of the Gepner model  $(2)^4 = \mathcal{C}_{1,0,0,1}$  which in the geometric interpretation  $(\Omega_X^0, \omega_1, V_{1,0,1}, B_{1,0,0})$  of Proposition 2.11 on the  $\mathbb{Z}_4$ -orbifold  $X_{1,0,1} = \widetilde{A_{1,0,1}/\mathbb{Z}_4}$  (with  $A_{1,0,1} = \mathbb{R}^4/\Lambda_{1,0,1}$ ,  $\Lambda_{1,0,1} = R_1\mathbb{Z}^2 \oplus R_2\mathbb{Z}^2$  as in (2.2),  $R_1 = R_2 = 1$ ) amount to deformations of the radii  $R_1, R_2$  of the torus  $A_{\alpha,\beta,\gamma}$  and of the B-field to  $B_{\alpha,\beta,\beta'}$  as in Proposition 2.11.*

For any refined geometric interpretation  $(\Omega, \omega, V, B)$  of the four-plane

$$x_{1,0,0,1} = \text{span}_{\mathbb{R}} \left( u_1 = \tilde{\Omega}_1, u_2 = \tilde{\Omega}_2, u_3 = \hat{v}_3^0 + \hat{v}_3, u_4 = 4\hat{v}^0 + \check{B}_4 + 5\hat{v} \right) \in \widetilde{\mathcal{M}}^{K3}$$

specifying  $\mathcal{C}_{1,0,0,1}$  in  $\mathcal{M}^{K3}$  as in Proposition 2.11, the following holds: If  $\Omega$  gives the complex structure of the Fermat quartic  $X_{\text{Fermat}} = X(f_0, f_0)$  as in (3.4), then

$$\begin{aligned} \Omega &= \Omega_{1,0,0,1}^X = \text{span}_{\mathbb{R}} \left( u_3 = \hat{v}_3^0 + \hat{v}_3, u_4 = 4\hat{v}^0 + \check{B}_4 + 5\hat{v} \right), \\ \mathcal{U}_X^0 &:= \text{span}_{\mathbb{R}} \left( \omega - \langle \omega, B \rangle v, v^0 + B + \left( V - \frac{1}{2} \langle B, B \rangle \right) v \right) = \text{span}_{\mathbb{R}} \left( \tilde{\Omega}_1, \tilde{\Omega}_2 \right) = \Omega_X^0, \end{aligned}$$

with  $v^0, v$  generators of  $H^0(X, \mathbb{Z})$  and  $H^4(X, \mathbb{Z})$  in this geometric interpretation. Furthermore, the fields  $V_{\pm}^{(1)}, V_{\pm}^{(2)}$  of (4.1) give deformations which leave invariant the two-plane  $\mathcal{U}_X^0$  that encodes the complexified Kähler structure of this geometric interpretation.

*Proof:*

The statements about the interpretation of deformations in terms of the  $\mathbb{Z}_4$ -orbifold construction follow from the above discussion, because solely the deformations listed are compatible with the  $\mathbb{Z}_4$ -action. Note that the induced deformation of the four-plane  $x_{1,0,0,1}$  leaves the two-plane  $\Omega_X^0 = \text{span}_{\mathbb{R}} \left( \tilde{\Omega}_1, \tilde{\Omega}_2 \right)$  invariant, as this plane is shared by all  $x_{\alpha, \beta, \beta', \gamma}$  according to Proposition 2.11. Moreover, by the same proposition the lattice  $\mathcal{Q} := x_{1,0,0,1} \cap H^{\text{even}}(X, \mathbb{Z})$  has rank four and is generated by the pairwise perpendicular lattice vectors  $u_1, u_2, u_3, u_4$  with

$$\langle u_1, u_1 \rangle = 2 = \langle u_2, u_2 \rangle, \quad \langle u_3, u_3 \rangle = 8 = \langle u_4, u_4 \rangle.$$

For the Fermat quartic  $X_{\text{Fermat}} = X(f_0, f_0)$  with complex structure  $\Omega_{\text{Fermat}} \subset H^2(X, \mathbb{R})$  by [52] the quadratic form associated to  $\Omega_{\text{Fermat}} \cap H^2(X, \mathbb{Z})$  (see Definition 3.3) is  $\text{diag}(8, 8)$ . However, the only primitive sublattice of  $\mathcal{Q}$  with this quadratic form is the one generated by  $u_3, u_4$ . It follows that for every refined geometric interpretation  $(\Omega, \omega, V, B)$  of  $\mathcal{C}_{1,0,0,1}$  with  $\Omega = \Omega_{\text{Fermat}}$ , the complex structure of the Fermat quartic, we must have  $\Omega_{\text{Fermat}} = \Omega_{1,0,0,1}^X$  as claimed and thus also  $\mathcal{U}_X^0 = \Omega_X^0$  as claimed. Above I have already argued that the deformations given by  $V_{\pm}^{(1)}, V_{\pm}^{(2)}$  of (4.1) leave the plane  $\mathcal{U}_X^0$  invariant, completing the proof.  $\square$

Note that the result of Proposition 4.2 does not imply that all refined geometric interpretations of  $\mathcal{C}_{1,0,0,1}$  with complex structure of the Fermat quartic agree: According to Definition 2.6, such a refined geometric interpretation is given by a decomposition  $x_{1,0,0,1} = \Omega \perp \mathcal{U}$  into perpendicular oriented two-planes together with an appropriate choice of null vectors  $v^0, v$  which by Lemma 2.7 allows to read off the data  $(\omega, V, B)$  from  $\mathcal{U}$ . However, given the decomposition  $x_{1,0,0,1} = \Omega_{1,0,0,1}^X \perp \mathcal{U}_X^0$ , an infinity of pairs of null vectors obeying conditions (1) and (2) of Definition 2.6 exists. Therefore I will not be able to show that the refined geometric interpretation of  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  on the quartic  $X(f_1, f_2)$  given in Lemma 3.2 and Proposition 2.11 agrees with the one claimed to exist in Result 3.1. The proof of Result 3.1 does not require such an identification.

Proposition 4.1 allows to study the model  $\mathcal{C}_{1,0,0,1} = (2)^4$  from a different perspective, namely as a model arising as orbifold of a certain Landau-Ginzburg model at criticality [80, 81]. This viewpoint, taken from [33], implies (see also [82, (74)])

**Fact 4.3** [28, 33]

The parameter space  $\widetilde{\mathcal{M}}^{K3}$  of SCFTs on  $K3$  contains a subspace of the form

$$\text{O}^+(2, 19; \mathbb{R})/\text{SO}(2) \times \text{O}(19) \quad \times \quad \text{O}^+(2, 1; \mathbb{R})/\text{SO}(2) \times \text{O}(1)$$

of SCFTs associated to quartic K3 surfaces in  $\mathbb{CP}^3$  with normalized Kähler class  $\omega = \omega_{FS}$ , the class of the Fubini-Study metric, and B-field  $B = b\omega_{FS}$  for some  $b \in \mathbb{R}$ . It is the space of models which arise as infrared fixed points of the renormalization group flow from linear sigma models in  $\mathbb{CP}^3$  according to [33]. The first factor of this space accounts for the choice of the complex structure of the quartic K3 surface, while the second factor captures the parameters  $V \in \mathbb{R}^+$  of the volume and  $b \in \mathbb{R}$  of the B-field.

Fixing the complex structure to that of the Fermat quartic  $X_{Fermat} = X(f_0, f_0)$  of (3.4) and identifying any two equivalent SCFTs in the resulting space, one obtains a space

$$\mathcal{M}^{Fermat} \cong \mathrm{O}^+(2, 1; \mathbb{Z}) \backslash \mathrm{O}^+(2, 1; \mathbb{R}) / \mathrm{SO}(2) \times \mathrm{O}(1) \cong \mathbb{S}^2 - \{\infty\}$$

of SCFTs associated to the Fermat quartic with normalized Kähler class  $\omega = \omega_{FS}$  and B-field  $B_{FS} = b\omega_{FS}$ ,  $b \in \mathbb{R}$ . It has two special points with non-trivial monodromy: There is one point with monodromy of order 2 where the SCFT description is expected to break down, while the second special point has monodromy of order 4 and gives the Gepner model  $(2)^4$ . The deformations of this model given by fields

$$\Phi_{\pm n_0, 0; \pm n_0, 0}^{n_0} \otimes \Phi_{\pm n_1, 0; \pm n_1, 0}^{n_1} \otimes \Phi_{\pm n_2, 0; \pm n_2, 0}^{n_2} \otimes \Phi_{\pm n_3, 0; \pm n_3, 0}^{n_3} \quad \text{with} \quad n_i \in \{0, 1, 2\}, \quad \sum_{i=0}^3 n_i = 4$$

amount to deformations of the defining polynomial of  $X(f_0, f_0)$  by monomials of the form  $\delta x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3}$ ,  $\delta \in \mathbb{C}$ .

It is not hard to translate Fact 4.3 into the language of our moduli space, i.e. to combine the results of [33] with those of [23]:

#### Corollary 4.4

The Gepner model  $(2)^4$  has a refined geometric interpretation  $(\Omega_{Fermat}, \omega_{FS}, V_{FS}, B_{FS})$  with  $\Omega_{Fermat}$  the complex structure of the Fermat quartic  $X(f_0, f_0) \subset \mathbb{CP}^3$  of (3.4),  $\omega_{FS}$  the Kähler class induced by the Fubini-Study metric in  $\mathbb{CP}^3$ ,  $V_{FS} = \frac{1}{2}$ , and  $B_{FS} = -\frac{1}{2}\omega_{FS}$ . Furthermore, the deformations  $V_{\pm}^{(1)}$ ,  $V_{\pm}^{(2)}$  of (4.1) give pure complex structure deformations within the family

$$X(f_1, f_2): f_1(x_0, x_1) + f_2(x_2, x_3) = 0 \quad \text{in} \quad \mathbb{CP}^3, \quad f_k(y_1, y_2) = y_1^4 + y_2^4 + \delta_k y_1^2 y_2^2, \quad \delta_k \in \mathbb{C}$$

in this geometric interpretation.

*Proof:*

Using Definition 2.6 and Lemma 2.7 one finds that the family of SCFTs  $\mathcal{M}^{Fermat}$  is given by four-planes  $x = \Omega_{Fermat} \perp \mathcal{U}_{V,b} \subset H^{even}(X, \mathbb{R})$  with fixed complex structure  $\Omega_{Fermat}$  of  $X_{Fermat}$  and complexified Kähler structure  $(\omega_{FS}, V, B = b\omega_{FS})$ ,  $V \in \mathbb{R}^+$ ,  $b \in \mathbb{R}$ , i.e. with

$$\mathcal{U}_{V,b} = \text{span}_{\mathbb{R}}(\omega_{FS} - 4bv_{FS}, v_{FS}^0 + b\omega_{FS} + (V - 2b^2)v_{FS}),$$

where  $v_{FS}^0$ ,  $v_{FS}$  generate  $H^0(X, \mathbb{Z})$  and  $H^4(X, \mathbb{Z})$ , respectively. It is convenient to introduce the complex parameter

$$\tau := b + i\sqrt{\frac{V}{2}} \in \mathbb{H},$$

and one finds that  $\tau, \tau' \in \mathbb{H}$  corresponding to four-planes  $x = \Omega_{Fermat} \perp \mathcal{U}_{V,b}$  and  $x' = \Omega_{Fermat} \perp \mathcal{U}_{V',b'}$  specify the same SCFT iff  $\tau = \gamma\tau'$  for some  $\gamma \in \Gamma_0(2)_+$ , the normalizer of  $\Gamma_0(2)$  in  $\mathrm{PSL}_2(\mathbb{R})$  [50, §3]. In other words,  $\mathcal{M}^{Fermat}$  of Fact 4.3 is given by  $\Gamma_0(2)_+ \backslash \mathbb{H} \cong$

$\mathbb{S}^2 - \{\infty\}$ . This space indeed has two special points with non-trivial stabilizer in  $\Gamma_0(2)_+$ , i.e. with non-trivial monodromy in  $\mathcal{M}^{Fermat}$ . One of these points has monodromy of order 4, namely  $\tau = -\frac{1}{2} + \frac{i}{2}$ , which according to Fact 4.3 gives the Gepner point. Hence  $(2)^4$  has refined geometric interpretation on the Fermat quartic with complexified Kähler structure encoded in  $\mathcal{U}_{\frac{1}{2}, -\frac{1}{2}}$ , amounting to  $V = \frac{1}{2}$ ,  $B = -\frac{1}{2}\omega_{FS}$  as claimed.

The claim about the deformations corresponding to  $V_{\pm}^{(1)}$ ,  $V_{\pm}^{(2)}$  of (4.1) follows directly from Fact 4.3 together with the result of Appendix A that in  $X(f_1, f_2)$  by appropriate coordinate transformations the polynomials  $f_1, f_2$  can be brought into the form  $f_k(y_1, y_2) = y_1^4 + y_2^4 + \delta_k y_1^2 y_2^2$  with  $\delta_k \in \mathbb{C}$ .  $\square$

As a compatibility check also note the following: By Proposition 4.1 we can construct the Gepner model  $(2)^4$  in terms of an orbifold of the toroidal model  $\mathcal{T}_{1,0,0,1}$  which can be carried out in two steps: With  $\iota$  the “Gepner orbifold” of (D.3),  $\mathcal{T}_{1,0,0,1}/\langle \iota^2 \rangle = (\hat{2})^4$  by the discussion in Appendix D.2, and then  $(\hat{2})^4/\langle \iota \rangle = (2)^4$ . This second orbifold has an “inverse” obtained by the action

$$\hat{\sigma}: \bigotimes_{j=1}^4 \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j} \longmapsto e^{\frac{2\pi i}{4}[(\overline{m}_1 + m_1) + (\overline{m}_2 + m_2)]} \bigotimes_{j=1}^4 \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j} \quad (4.2)$$

on the fields of  $(2)^4$ . Together with the identifications made in Fact 4.3 this action descends to the automorphism  $\sigma$  of (3.2) which was used in Inose’s construction. In other words,  $(2)^4/\langle \hat{\sigma} \rangle = (\hat{2})^4 = (2)^2 \otimes (2)^2/\langle \iota^2 \rangle$  is indeed a lift of Inose’s construction to the SCFT level.

## 4.2 The proof

To complete the proof of Result 3.1 let me first take stock of what we have achieved so far. By Corollary 4.4 the claim is true for the special SCFT  $\mathcal{C}_{1,0,0,1}$  which agrees with the Gepner model  $(2)^4$  by Proposition 4.1. By Proposition 4.2 this implies that for the four-plane  $x_{1,0,0,1} \in \widetilde{\mathcal{M}}^{K3}$  specifying this model within the moduli space,  $x_{1,0,0,1} = \Omega_{1,0,0,1}^X \perp \mathcal{U}_X^0$  with  $\Omega_{1,0,0,1}^X = \Omega_{Fermat}^X$  (notations as in Propositions 2.11, 4.2, and Corollary 4.4). Moreover, the variation of the parameters  $\alpha, \beta, \beta', \gamma$  in  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  away from  $(\alpha, \beta, \beta', \gamma) = (1, 0, 0, 1)$  leave the two-plane  $\mathcal{U}_X^0$  invariant,  $x_{\alpha, \beta, \beta', \gamma} = \Omega_{\alpha, \beta, \beta', \gamma}^X \perp \mathcal{U}_X^0$ . Finally, by Lemma 3.2 any refined geometric interpretation of  $x_{\alpha, \beta, \beta', \gamma}$  using  $\Omega_{\alpha, \beta, \beta', \gamma}^X$  to specify the complex structure gives the complex structure of the quartic  $X(f_1, f_2)$  obtained from  $\alpha, \beta, \beta', \gamma$  as in Result 3.1. Recall from the discussion at the end of Section 2.1 that a decomposition of a four-plane  $x \in \widetilde{\mathcal{M}}^{K3}$  into two oriented two-planes  $x = \Omega \perp \mathcal{U}$  with choice of ordering amounts to an interpretation of the corresponding SCFT on  $K3$  in terms of a generalized  $K3$  structure. Hence the above already shows that  $\mathcal{C}_{\alpha, \beta, \beta', \gamma}$  can be interpreted in terms of generalized  $K3$  structures in accord with the claim of Result 3.1. However, the claim made there is stronger in that it refers to a refined geometric interpretation rather than a generalized  $K3$  structure. See also the end of Section 2.1 for a discussion of this distinction: It remains to show that the null vectors  $v_{FS}^0, v_{FS}$  needed for the refined geometric interpretation of  $\mathcal{C}_{1,0,0,1}$  in terms of the Fermat quartic with normalized Kähler class  $\omega_{FS}$ , volume  $V_{FS} = \frac{1}{2}$ , and B-field  $B_{FS} = -\frac{1}{2}\omega_{FS}$  (Corollary 4.4) are compatible with interpreting  $\Omega_{\alpha, \beta, \beta', \gamma}^X$  in  $x_{\alpha, \beta, \beta', \gamma} = \Omega_{\alpha, \beta, \beta', \gamma}^X \perp \mathcal{U}_X^0$  as two-plane yielding a complex structure for all admissible  $\alpha, \beta, \beta', \gamma$ . In other words, we need to show that  $\Omega_{\alpha, \beta, \beta', \gamma}^X \perp v_{FS}^0$  and  $\Omega_{\alpha, \beta, \beta', \gamma}^X \perp v_{FS}$  for all admissible  $\alpha, \beta, \beta', \gamma$ .

This follows by means of the identifications of deformations of  $\mathcal{C}_{1,0,0,1}$  that I have given in Section 4.1: By Proposition 4.2, the fields  $V_{\pm}^{(1)}$  and  $V_{\pm}^{(2)}$  of (4.1) give the deformations of

$x_{1,0,0,1}$  into the four parameter family  $x_{\alpha,\beta,\beta',\gamma}$ . On the other hand, by Corollary 4.4 within the refined geometric interpretation ( $\Omega_{Fermat} = \Omega_{1,0,0,1}^X, \omega_{FS}, V_{FS}, B_{FS}$ ) of  $\mathcal{C}_{1,0,0,1}$  these fields induce pure complex structure deformations of  $X_{Fermat} = X(f_0, f_0)$  to  $X(f_1, f_2)$  with  $f_k(y_1, y_2) = y_1^4 + y_2^4 + \delta_k y_1^2 y_2^2$ ,  $\delta_k \in \mathbb{C}$ . This amounts to  $v_{FS}^0, v_{FS} \perp \Omega_{\alpha,\beta,\beta',\gamma}^X$  as needed.  $\square$

The use of the results of [33] in the above proof ties this work to seminal insights from the physics literature. However, one would hope to be able to find a proof completely within the language of algebraic geometry instead of having to mix two viewpoints. A possible strategy for such a proof involves a more detailed study of the model  $(\widehat{2})^4$  (see Appendix D.2) and its two refined geometric interpretations induced from its two  $\mathbb{Z}_2$ -orbifold constructions: One arising from the Kummer construction for the standard torus  $A_{1,0,1} = \mathbb{R}^4/\mathbb{Z}^4$  with vanishing B-field,  $(\widehat{2})^4 = \mathcal{T}_{1,0,0,1}/\mathbb{Z}_2 = (2)^2 \otimes (2)^2/\langle \iota^2 \rangle$  with  $\iota$  as in (D.3), and the other as  $\mathbb{Z}_2$ -orbifold CFT arising from the extension of the orbifold construction  $X(f_1, f_2)/\langle \sigma \rangle$  to SCFT level,  $(\widehat{2})^4 = (2)^4/\langle \widehat{\sigma} \rangle$  with  $\sigma, \widehat{\sigma}$  as in (3.2), (4.2), respectively. One should find the appropriate lattice automorphism of  $H^{even}(X, \mathbb{Z})$  which relates these two refined geometric interpretations of the relevant four-plane  $x_{(\widehat{2})^4} \in \widehat{\mathcal{M}}^{K3}$  to one another. Starting from the Kummer construction the resulting normalized Kähler class in the geometric interpretation on  $X(f_1, f_2)/\langle \sigma \rangle$  needs to be characterized by its intersection numbers with all other two-cycles in  $X(f_1, f_2)/\langle \sigma \rangle$  (or otherwise) to show that it agrees with the class of the orbifold limit of an Einstein metric descending from  $\omega_{FS}$  on  $X(f_1, f_2) \subset \mathbb{CP}^3$ , the class induced by the Fubini-Study metric on  $\mathbb{CP}^3$ . The result of Proposition 3.5 was obtained as a welcome side effect of my quest for such a proof.

## 5 Discussion

This work aims to provide a self-contained description of how to construct SCFTs  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  associated to the smooth quartic  $K3$  surfaces

$$X(f_1, f_2): \quad f_1(x_0, x_1) + f_2(x_2, x_3) = 0 \quad \text{in} \quad \mathbb{CP}^3 \quad (5.1)$$

with normalized Kähler class  $\omega_{FS}$  induced by the Fubini-Study metric on  $\mathbb{CP}^3$ , volume  $V = \frac{1}{2}$ , and B-field  $B = -\frac{1}{2}\omega_{FS}$ . The construction itself is simple, since  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  turns out to be a standard  $\mathbb{Z}_4$ -orbifold of a toroidal SCFT. I regard this as a virtue rather than a disadvantage, since it implies that the family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  does not only lend itself to all field theory techniques that are linked to the algebraic description through (5.1) but also that the underlying vertex operator algebras are completely explicitly accessible. Furthermore  $\mathcal{C}_{1,0,0,1}$  agrees with the  $(2)^4$  Gepner model, such that the family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  can be viewed as a deformation of that model. Altogether the four-parameter family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  is well under control, both from a SCFT and an algebraic point of view, and as such it is the first known example of its kind.

My construction can be viewed as a generalization to SCFTs of a classical construction by Inose [52] by employing a crude version of mirror symmetry. As a by-product, motivated by discussions with M. Headrick and T. Wiseman, a characterization in terms of a Kummer construction is obtained for the Kähler class induced by the class of the Fubini Study metric on an orbifold of, say, the Fermat quartic. This makes the Kähler-Einstein metric in the former class accessible to numerical approaches developed in [53]. One may hope that such numerical approaches can be generalized to the level of SCFT to begin an analysis of as yet unexplored SCFTs which have no orbifold description.



Rational SCFTs seem not to play a central rôle within the family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$ . While not all theories with  $\alpha, \beta, \beta', \gamma \in \mathbb{Q}$  are rational, these are the parameter values at which the corresponding quartic hypersurfaces (5.1) are “very attractive”, i.e. they have maximal Picard number. It would be interesting to know whether any particular intrinsic property of the underlying SCFTs distinguishes rational from non-rational values of  $\alpha, \beta, \beta', \gamma$ . After all, within  $\mathcal{M}^{K3}$  these theories are characterized by the fact that the four-plane  $x_{\alpha,\beta,\beta',\gamma} \subset H^{even}(X, \mathbb{R})$  is generated by lattice vectors in  $H^{even}(X, \mathbb{Z})$ .

The proof for the main result of this work links my construction to Witten’s results on gauged linear sigma models [33]. An independent proof would be desirable, but this link could be of considerable use in applications: Although the relation between Landau-Ginzburg models and SCFTs has been known for a long time [80, 81], this connection has only rarely been put to use in SCFT. Recent exceptions to this rule are novel techniques to construct D-branes by using matrix factorizations, where by an unpublished result of Kontsevich topological D-branes in Landau-Ginzburg models are classified in terms of matrix factorizations [83] and therefore are expected to translate to boundary states in SCFT [36, 37, 38, 39, 40, 41, 42, 43, 44]. While for supersymmetric minimal models this correspondence is fully confirmed and understood, for Gepner models a number of problems remain open. E.g. a special class of matrix factorizations is expected to correspond to arbitrary permutation branes [84, 39, 41, 44], but the full correspondence is not yet established. The family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  studied in the present work seems to provide a promising testing ground for these methods: Its algebraic description is tailor made for a study in the language of Landau-Ginzburg models, while its SCFT construction makes it accessible to all techniques provided by representation theory. Moreover, since  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  is a family of deformations of the Gepner model  $(2)^4$ , such a study would surpass known results. Very recently a step in this direction has been carried out in [85]. There the model  $(2) \otimes (2)$  is investigated which can be viewed as a  $\mathbb{Z}_4$ -orbifold of the Gepner model  $(2)^2$ ; note  $(2)^2 \otimes (2)^2 / \mathbb{Z}_4 = (2)^4$ .

While a large part of the tool-set used for the proof of my main result relies on the particularities of SCFTs associated to  $K3$ , above all on the high amount of supersymmetry which these models enjoy, insights into techniques like matrix factorization or the chiral de Rham complex as briefly mentioned in the Introduction can be hoped to generalize to higher dimensions. Indeed, all these applications intrinsically use a description of the relevant SCFTs in terms of  $N = (2, 2)$  supersymmetry. In the geometric interpretation of  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  this corresponds to the fact that I explicitly determine a complex structure for the underlying  $K3$  surfaces. From this viewpoint the family  $\mathcal{C}_{\alpha,\beta,\beta',\gamma}$  is special solely because we have several useful descriptions for it, not because its target space has complex dimension 2, and it should be possible to take profit from these descriptions which can be hoped to generalize to higher dimensions.

## Appendix A Quartic representation of elliptic curves

Consider an elliptic curve in Weierstraß form (1.6). To express this curve within  $\mathbb{CP}_{2,1,1}$ , factorize the right hand side of (1.6),

$$y^2 t = \prod_{i=1}^3 (x - \xi_i t) \iff (ty)^2 = t \prod_{i=1}^3 (x - \xi_i t) \quad \text{if } t \neq 0.$$

Set  $y_0 = ty$  and with suitable  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  let  $t = \alpha y_1 + \beta y_2$ ,  $x = \gamma y_1 + \delta y_2$  to obtain an equation

$$E_f : y_0^2 = f(y_1, y_2) \quad \text{in } \mathbb{CP}_{2,1,1}$$

with  $f$  a homogeneous polynomial of degree 4.

As a helpful example consider the elliptic curve with period  $\tau = i$ . Its  $j$ -invariant is well-known,  $j(i) = 1728$ , so its Weierstraß form can be taken as

$$y^2 t = x(x - t)(x + t).$$

Let  $\varepsilon$  denote a primitive eighth root of unity,  $\lambda \in \mathbb{C}$  such that  $\lambda^{-3} = 2i$ , and set  $ty = y_0$ ,  $t = \lambda(y_1 - \varepsilon y_2)$ ,  $x = -i\lambda(y_1 + \varepsilon y_2)$  as above. This yields

$$\begin{aligned} y_0^2 &= \lambda^3 (y_1 - \varepsilon y_2) i (y_1 + \varepsilon y_2) (1 + i) (y_1 + i\varepsilon y_2) (1 - i) (y_1 - i\varepsilon y_2) \\ &= y_1^4 + y_2^4 =: f_0(y_1, y_2). \end{aligned}$$

In general for non-degenerate elliptic curves we can assume without loss of generality that  $f$  has the form

$$f(y_1, y_2) = y_1^4 + 2\kappa y_1^2 y_2^2 + y_2^4, \quad \kappa \in \mathbb{C}.$$

Indeed, one first finds  $\alpha, \beta, \gamma, \delta$  above such that  $f(y_1, y_2) = \nu_1 y_1^4 + 2\kappa' y_1^2 y_2^2 + \nu_2 y_2^4$ : Assuming  $\alpha\beta\gamma\delta = 1$  with  $A := \alpha\beta$ ,  $B := \alpha\delta$  and inserting  $t = \alpha y_1 + \beta y_2$ ,  $x = \gamma y_1 + \delta y_2$  directly into (1.6) one needs to solve

$$\begin{aligned} 0 &= B^2 A^{-1} + 3A^{-1} - 27a (B^{-2} A + 3A) - 216b A^2 B^{-1}, \\ 0 &= B^{-2} A^{-1} + 3A^{-1} - 27a (B^2 A + 3A) - 216b A^2 B. \end{aligned}$$

The matrix with coefficients  $\alpha, \beta, \gamma, \delta$  needs to be invertible, which implies  $B^2 \neq 1$ . Hence we can divide by  $(B - B^{-1})$ , and setting  $C := B + B^{-1}$  the above system of equations is equivalent to

$$\begin{aligned} D &= 27a A^2, \\ 0 &= (1 + D) C - 216b A^3, \\ 0 &= (1 - D) C^2 - 216b A^3 C + 4(1 - D). \end{aligned}$$

This system can be solved in terms of a quartic equation for  $D$ . Having brought  $f$  to the form  $\nu_1 y_1^4 + 2\kappa' y_1^2 y_2^2 + \nu_2 y_2^4$ , where non-degeneracy implies  $\nu_1, \nu_2 \neq 0$ , one merely needs to rescale the  $y_k$  to obtain the desired form  $y_1^4 + 2\kappa y_1^2 y_2^2 + y_2^4$ .

To determine under which circumstances two different values of  $\kappa \in \mathbb{C}$  in  $f(y_1, y_2) = y_1^4 + 2\kappa y_1^2 y_2^2 + y_2^4$  yield the same elliptic curve first restrict to  $\Im(\kappa) \geq 0$  by employing  $(y_1, y_2) \mapsto (iy_1, y_2)$ . For real  $\kappa$  one can furthermore assume  $\kappa \geq 0$ . By a similar calculation to the above one finds that  $\kappa$  and  $\kappa'$  with non-negative imaginary parts yield the same elliptic curve iff  $\kappa' = \frac{\kappa+3}{1-\kappa}$  or  $\kappa = \frac{\kappa'+3}{1-\kappa'}$ . In other words, a fundamental domain for  $\kappa$  is  $\Gamma \setminus \{z \in \mathbb{C} \mid z \sim -z\}$  where  $\Gamma \subset \text{PSL}_2(\mathbb{R})$  acts by Möbius transforms and is generated by  $\kappa \mapsto \frac{\kappa+3}{1-\kappa}$ . This transformation has order 3 and correctly identifies the three values  $\kappa \in \{\pm 1, \infty\}$  for which the elliptic curve degenerates. Its unique fixed point with non-negative imaginary part is  $\kappa = i\sqrt{3}$ . The circle about  $\kappa = 1$  of radius 2 contains both  $\kappa = -1$  and  $\kappa = i\sqrt{3}$ . Hence a fundamental domain for those  $\kappa$  which yield non-degenerate elliptic curves is bounded by the interval  $(-1, 1)$  on the real axis together with the two circle arcs  $|\kappa \pm 1| = 2$  between the real axis and  $\kappa = i\sqrt{3}$ . These latter two arcs are glued together, while on  $(-1, 1)$  we impose  $z \sim -z$ . Summarizing one obtains (1.7), as depicted in Figure 1.1.

## Appendix B Minimal models

Let me recall the construction of the  $N = (2, 2)$  superconformal minimal models [86, 87, 88, 89]. In fact we will only be concerned with the so-called  $A$ -series of minimal models, so by abuse of notation I use  $(k)$  for  $k \in \mathbb{N}$  to denote the coset model

$$\frac{\mathrm{SU}(2)_k \otimes \mathrm{U}(1)_2}{\mathrm{U}(1)_{k+2, \text{diag}}}$$

at central charges

$$c = \bar{c} = \frac{3k}{k+2}$$

[86, 87, 90, 91]. Following [92, 93] I use the most convenient description of the field content of  $(k)$  in terms of a free boson  $\varphi$  and the parafermion model at level  $k$  found in [88, 94]: Let  $\psi_l$ ,  $l \in \{1, \dots, k-1\}$  denote the  $\mathbb{Z}_k$  parafermion algebra, i.e.

$$\psi_l(z)\psi_{l'}(w) \sim \begin{cases} c_{l,l'}(z-w)^{-2l'/k} (\psi_{l+l'}(w) + \dots) & \text{if } l+l' < k, \\ c_{l,l'}(z-w)^{-2l'/k} (\psi_{l+l'-k}(w) + \dots) & \text{if } l+l' > k, \\ (z-w)^{-2l'/k} (\mathbb{1} + c_k(z-w)^2 T_{pf}(w) + \dots) & \text{if } l+l' = k, \end{cases}$$

where  $T_{pf}$  is the Virasoro field of the parafermion model. I denote by  $\xi_{m,\bar{m}}^l$  the primary fields of the parafermion theory,  $l \in \{0, \dots, k\}$ ,  $m, \bar{m} \in \mathbb{Z}/2(k+2)\mathbb{Z}$ . In particular,  $\psi_l = \xi_{2l,0}^0$ . The primary fields  $\Phi_{m,s;\bar{m},\bar{s}}^l(z, \bar{z}) = \psi_{m,s}^l(z) \otimes \psi_{\bar{m},\bar{s}}^l(\bar{z})$  of the minimal model  $(k)$  can then be expressed as follows:

for  $l \in \{0, \dots, k\}$ ,  $m, \bar{m} \in \mathbb{Z}/2(k+2)\mathbb{Z}$ ,  $s, \bar{s} \in \mathbb{Z}/4\mathbb{Z}$ ,  $l+m+s \equiv \bar{l}+\bar{m}+\bar{s} \equiv 0 \pmod{2}$ :

$$\begin{aligned} \Phi_{m,s;\bar{m},\bar{s}}^l(z) &= \xi_{m-s,\bar{m}-\bar{s}}^l e^{-i\beta_k Q_{m,s}\varphi(z) - i\beta_k Q_{\bar{m},\bar{s}}\bar{\varphi}(\bar{z})}, \\ \text{where } \beta_k &:= \sqrt{\frac{k+2}{k}}, \quad Q_{m,s} := \frac{m}{k+2} - \frac{s}{2}. \end{aligned}$$

I use the same symbols  $\Phi_{m,s;\bar{m},\bar{s}}^l$  to label the conformal families of these primary fields. More precisely,  $[\Phi_{m,s;\bar{m},\bar{s}}^l]$  denotes the representation built on the primary  $\Phi_{m,s;\bar{m},\bar{s}}^l$  with respect to the bosonic subalgebra of the superconformal algebra. One finds the fusion rules

$$\text{for } \Phi_{m,s;\bar{m},\bar{s}}^l(z, \bar{z}) = \psi_{m,s}^l(z) \otimes \psi_{\bar{m},\bar{s}}^l(\bar{z}) : \quad \left[ \psi_{m,s}^l \right] \times \left[ \psi_{m',s'}^{l'} \right] = \sum_{\substack{\bar{l}=|l-l'| \\ \bar{l} \equiv l+l' \pmod{2}}}^{\min(l+l', 2k-l-l')} \left[ \psi_{m+m',s+s'}^{\bar{l}} \right], \quad (\text{B.1})$$

which enjoy the following  $\mathbb{Z}_{k+2}$  symmetry:

$$\Phi_{m,s;\bar{m},\bar{s}}^l(z) \longmapsto e^{\frac{2\pi i}{2(k+2)}(m+\bar{m})} \cdot \Phi_{m,s;\bar{m},\bar{s}}^l(z). \quad (\text{B.2})$$

The left-handed superconformal algebra is generated by

$$j(z) = \frac{i}{\beta_k} \partial \varphi(z), \quad G^+(z) = \frac{1}{\beta_k} \psi_1(z) e^{i\beta_k \varphi(z)}, \quad G^-(z) = \frac{1}{\beta_k} \psi_{k-1}(z) e^{-i\beta_k \varphi(z)},$$

and analogously on the right hand side, i.e.  $G^+$  and  $G^-$  belong to the same conformal family  $[\Phi_{0,2;0,0}^0]$ , where it should be kept in mind that  $\Phi_{0,2;0,0}^0$  is primary only with respect to the bosonic subalgebra of the superconformal algebra, as mentioned above. Moreover, up to

shifts by even integers,  $Q_{m,s}$  is the charge of  $\Phi_{m,s;\overline{m},\overline{s}}^l$  with respect to the  $u(1)$  current  $j$  of the superconformal algebra. All charges  $Q$  of primaries  $\Phi_{m,s;\overline{m},\overline{s}}^l$  obey  $|Q| \leq 1$ . One has

$$\Phi_{m,s;\overline{m},\overline{s}}^l = \Phi_{m+k+2,s+2;\overline{m}+k+2,\overline{s}+2}^{k-l},$$

and moreover,

$$\text{when } |m-s| \leq l: \quad h_{m,s}^l := \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8}$$

gives the conformal dimensions  $h_{m,s}^l, h_{\overline{m},\overline{s}}^l$  of  $\Phi_{m,s;\overline{m},\overline{s}}^l$ . The above formula holds in general up to shifts by integers. If neither  $\Phi_{m,s;\overline{m},\overline{s}}^l$  nor  $\Phi_{m+k+2,s+2;\overline{m}+k+2,\overline{s}+2}^{k-l}$  lie in the regime where the formula holds precisely, then one uses it for the representative with  $m-s = l-2$ , or, if this does not exist, for the one with  $m-s = l+2$ , and adds 1 to the result [95].

As mentioned above,  $(k)$  denotes the  $A$ -model at level  $k$ , i.e. this theory has primaries  $\Phi_{m,s;\overline{m},\overline{s}}^l$ , where  $s \equiv \overline{s} \bmod 2$ . Fields with even  $s$  live in the Neveu-Schwarz sector, while fields with odd  $s$  live in the Ramond sector. Moreover, fields with  $s - \overline{s} \equiv 0 \bmod 4$  are bosonic, while those with  $s - \overline{s} \equiv 2 \bmod 4$  are fermionic. Equivalently and more conveniently, a field is bosonic iff its left and right handed charges differ by an even integer. In particular,

$$\Phi_{m_1,s_1;\overline{m}_1,\overline{s}_1}^{l_1} \circ \Phi_{m_2,s_2;\overline{m}_2,\overline{s}_2}^{l_2} = (-1)^{\frac{1}{4}(s_1-\overline{s}_1)(s_2-\overline{s}_2)} \Phi_{m_2,s_2;\overline{m}_2,\overline{s}_2}^{l_2} \circ \Phi_{m_1,s_1;\overline{m}_1,\overline{s}_1}^{l_1}.$$

For example, at level  $k = 2$  one gets the following values for the conformal dimensions and charges  $(h_{m,s}^l, Q_{m,s})$  of the primary bosonic fields in the Neveu-Schwarz sector:

$\begin{smallmatrix} \backslash l \\ m \end{smallmatrix}$	$\begin{smallmatrix} 0, \\ s=0 \end{smallmatrix}$	$\begin{smallmatrix} 0, \\ s=2 \end{smallmatrix}$	$\begin{smallmatrix} 1, \\ s=0 \end{smallmatrix}$	$\begin{smallmatrix} 1, \\ s=2 \end{smallmatrix}$	$\begin{smallmatrix} 2, \\ s=0 \end{smallmatrix}$	$\begin{smallmatrix} 2, \\ s=2 \end{smallmatrix}$
-3			$\left(\frac{5}{8}, -\frac{3}{4}\right)$	$\left(\frac{1}{8}, \frac{1}{4}\right)$		
-2	$\left(\frac{3}{4}, -\frac{1}{2}\right)$	$\left(\frac{1}{4}, \frac{1}{2}\right)$			$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\left(\frac{3}{4}, \frac{1}{2}\right)$
-1			$\left(\frac{1}{8}, -\frac{1}{4}\right)$	$\left(\frac{5}{8}, \frac{3}{4}\right)$		
0	$(0,0)$	$\left(\frac{3}{2}, \pm 1\right)$			$\left(\frac{1}{2}, 0\right)$	$(1, \pm 1)$
1			$\left(\frac{1}{8}, \frac{1}{4}\right)$	$\left(\frac{5}{8}, -\frac{3}{4}\right)$		
2	$\left(\frac{3}{4}, \frac{1}{2}\right)$	$\left(\frac{1}{4}, -\frac{1}{2}\right)$			$\left(\frac{1}{4}, \frac{1}{2}\right)$	$\left(\frac{3}{4}, -\frac{1}{2}\right)$
3			$\left(\frac{5}{8}, \frac{3}{4}\right)$	$\left(\frac{1}{8}, -\frac{1}{4}\right)$		
4	$(1, \pm 1)$	$\left(\frac{1}{2}, 0\right)$			$\left(\frac{3}{2}, \pm 1\right)$	$(0,0)$

(B.3)

The character of the conformal family  $\Phi_{m,s;\overline{m},\overline{s}}^l$  is given by

$$X_{m,s;\overline{m},\overline{s}}^l(\tau', z) = \chi_{m,s}^l(\tau', z) \cdot \chi_{\overline{m},\overline{s}}^l(\overline{\tau}', \overline{z}),$$

$$\chi_{m,s}^l(\tau', z) = \sum_{j=1}^k c_{4j+s-m}^l(\tau') \Theta_{2m-(k+2)(4j+s), 2k(k+2)} \left( \tau', \frac{z}{k+2} \right), \quad (\text{B.4})$$

with  $\tau' \in \mathbb{H}$ ,  $z \in \mathbb{C}$ , and where  $c_j^l, l \in \{0, \dots, k\}, j \in \mathbb{Z}/2k\mathbb{Z}$  are the level  $k$  string functions of  $\text{SU}(2)_k$ , and  $\Theta_{a,b}, a \in \mathbb{Z}/2b\mathbb{Z}$  denote classical theta functions of level  $b \in \mathbb{N}$  [29, 96, 89]. All minimal models are invariant under simultaneous left and right handed spectral flow, where the simple current  $\Phi_{1,1;0,0}^0$  is the operator which generates the spectral flow on the left. Hence the Neveu-Schwarz part of the partition function is given by

$$Z_{NS}(\tau', z) = \frac{1}{2} \sum_{\substack{l=0, \dots, k \\ m=-k-1, \dots, k+2, \\ l+m \equiv 0(2)}} \left( \chi_m^{l,0}(\tau', z) + \chi_m^{l,2}(\tau', z) \right) \left( \chi_m^{l,0}(\bar{\tau}', \bar{z}) + \chi_m^{l,2}(\bar{\tau}', \bar{z}) \right),$$

while the partition functions for the remaining sectors can be obtained from  $Z_{NS}$  by means of spectral flow.

## Appendix C Gepner models

In the main body of this paper I study SCFTs which can be viewed as internal theories of type IIA string theories. Gepner models [28, 29, 97] are heterotic string theories, which are obtained from certain type IIA models by a trick called heterosis. However, by abuse of terminology I instead call the internal parts of these type IIA theories Gepner models. To construct these models one first forms the fermionic tensor product of a number of  $N = (2, 2)$  minimal models  $(k_1), \dots, (k_r)$  as discussed in Appendix B, i.e. the NS and the R sectors are tensorized separately to obtain  $(k_1) \otimes \dots \otimes (k_r)$ . For the construction to work one needs to ensure that the total central charge of this model is a multiple of 3,

$$c = \bar{c} = \sum_{i=1}^r \frac{3k_i}{k_i + 2} = 3D, \quad D \in \mathbb{N}.$$

Our model  $(k_1) \otimes \dots \otimes (k_r)$  enjoys a cyclic symmetry  $\mathbb{Z}_M$  induced by (B.2) with  $M = \text{lcm}\{k_i + 2, i = 1, \dots, r\}$ . The symmetry is generated by

$$\zeta_M: \bigotimes_{j=1}^r \Phi_{m_j, s_j; m_j, \bar{s}_j}^{l_j} \longmapsto e^{\frac{2\pi i}{6}(cs_1)} \left( \prod_{j=1}^r e^{\frac{2\pi i}{2(k_j+2)}(m_j+m_j)} \right) \bigotimes_{j=1}^r \Phi_{m_j, s_j; m_j, \bar{s}_j}^{l_j}. \quad (\text{C.1})$$

The Gepner model  $(k_1) \dots (k_r)$  is the orbifold of  $(k_1) \otimes \dots \otimes (k_r)$  by this symmetry. For calculations it is useful to note that the  $\mathbb{Z}_M$  invariant part of  $(k_1) \otimes \dots \otimes (k_r)$  is given by those NS states with integral left and right handed charges and those R states with integral (half integral) left and right handed charges if  $D$  is even (odd). Moreover, the operator of two-fold left handed spectral flow,

$$U := \bigotimes_{j=1}^r \Phi_{2,2;0,0}^0,$$

in this orbifold maps the sector twisted by  $\zeta_M^m$  to the one twisted by  $\zeta_M^{m+1}$ . In other words,  $(k_1) \dots (k_r)$  is obtained from  $(k_1) \otimes \dots \otimes (k_r)$  by projecting onto those states with the correct charges and then generating all remaining states by repeated action of the two-fold left

handed spectral flow  $U$ . This process is also known as GSO projection or as Gepner's  $\beta$  method. Note that  $U$  has  $u(1)$  charge  $(-D)$ , so that our condition  $D \in \mathbb{N}$  ensures that all  $u(1)$  charges in a Gepner model are integral in the NS sector and integral or half integral in the R sector. Moreover, (B.1) shows that  $U$  is a simple current,

$$[U] \times \left[ \bigotimes_{j=1}^r \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j} \right] = \left[ \bigotimes_{j=1}^r \Phi_{m_j+2, s_j+2; \overline{m}_j+2, \overline{s}_j+2}^{l_j} \right]. \quad (C.2)$$

Note that the bosonic fields in a Gepner model are precisely those fields whose left and right handed charges differ by an even integer.

The Gepner model  $(k_1) \cdots (k_r)$  enjoys many symmetries, in particular phase symmetries inherited from (B.2),

$$\begin{aligned} \text{for } a_j \in \mathbb{Z}, \quad [a_1, \dots, a_r]: \quad & \bigotimes_{j=1}^r \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j} \\ & \longmapsto e^{\frac{2\pi i}{6}(cs_1)} \left( \prod_{j=1}^r e^{\frac{2\pi i}{2(k_j+2)} a_j (m_j + \overline{m}_j)} \right) \bigotimes_{j=1}^r \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j}. \end{aligned} \quad (C.3)$$

As an example, to calculate the partition function of the Gepner model  $(2)^2$  one uses the characters as obtained from (B.4), which with  $y = e^{2\pi i z}$  yield

$$\begin{aligned} (\chi_{0,0}^0 + \chi_{0,2}^0)(\tau', z) &= \frac{1}{2\eta(\tau')} \left( \sqrt{\frac{\vartheta_3(\tau', 0)}{\eta(\tau')}} \vartheta_3(2\tau', z) + \sqrt{\frac{\vartheta_4(\tau', 0)}{\eta(\tau')}} \vartheta_4(2\tau', z) \right), \\ (\chi_{-2,0}^0 + \chi_{-2,2}^0)(\tau', z) &= \frac{1}{2\eta(\tau')} \left( \sqrt{\frac{\vartheta_3(\tau', 0)}{\eta(\tau')}} \vartheta_2(2\tau', z) + \sqrt{\frac{\vartheta_4(\tau', 0)}{\eta(\tau')}} i\vartheta_1(2\tau', z) \right), \\ (\chi_{4,0}^0 + \chi_{4,2}^0)(\tau', z) &= \frac{1}{2\eta(\tau')} \left( \sqrt{\frac{\vartheta_3(\tau', 0)}{\eta(\tau')}} \vartheta_3(2\tau', z) - \sqrt{\frac{\vartheta_4(\tau', 0)}{\eta(\tau')}} \vartheta_4(2\tau', z) \right), \\ (\chi_{2,0}^0 + \chi_{2,2}^0)(\tau', z) &= \frac{1}{2\eta(\tau')} \left( \sqrt{\frac{\vartheta_3(\tau', 0)}{\eta(\tau')}} \vartheta_2(2\tau', z) - \sqrt{\frac{\vartheta_4(\tau', 0)}{\eta(\tau')}} i\vartheta_1(2\tau', z) \right), \\ (\chi_{1,0}^1 + \chi_{1,2}^1)(\tau', z) &= \frac{q^{\frac{1}{16}} y^{\frac{1}{4}}}{2\eta(\tau')} \sqrt{\frac{\vartheta_2(\tau', 0)}{\eta(\tau')}} \vartheta_3(2\tau', z + \frac{\tau'}{2}), \\ (\chi_{3,0}^1 + \chi_{3,2}^1)(\tau', z) &= \frac{q^{\frac{1}{16}} y^{\frac{1}{4}}}{2\eta(\tau')} \sqrt{\frac{\vartheta_2(\tau', 0)}{\eta(\tau')}} \vartheta_2(2\tau', z + \frac{\tau'}{2}). \end{aligned}$$

For the partition function of  $(2)^2$  with some patience from this one obtains

$$Z_{NS}^{(2)^2}(\tau', z) = \frac{1}{2} \left[ \left| \frac{\vartheta_2(\tau', 0)}{\eta(\tau')} \right|^4 + \left| \frac{\vartheta_3(\tau', 0)}{\eta(\tau')} \right|^4 + \left| \frac{\vartheta_4(\tau', 0)}{\eta(\tau')} \right|^4 \right] \left| \frac{\vartheta_3(\tau', z)}{\eta(\tau')} \right|^2. \quad (C.4)$$

## Appendix D The Gepner models $(2)^2$ , $(2)^4$ , and $(\widehat{2})^4$

### Appendix D.1 The Gepner model $(2)^2$

The partition function (C.4) of  $(2)^2$  has the form (1.4) of the partition function of a toroidal SCFT. Indeed, every  $N = (2, 2)$  SCFT at central charges  $c = \bar{c} = 3D$  with  $D \in \mathbb{N}$  which is invariant under spectral flow and only has integral  $\mathfrak{u}(1)$  charges in the NS sector with respect to the  $\mathfrak{u}(1)$  currents of the left and right handed superconformal algebras is expected to have a non-linear sigma model description on a Calabi-Yau manifold of complex dimension  $D$ . Moreover, if for the Gepner model  $(k_1) \cdots (k_r)$  one has  $r \leq D + 2$ , then this model is expected to have a non-linear sigma model realization on the Calabi-Yau hypersurface

$$z_1^{2+k_1} + \cdots + z_{D+2}^{2+k_{D+2}} = 0 \quad \text{in } \mathbb{CP}^{\frac{M}{2+k_1}, \dots, \frac{M}{2+k_{D+2}}},$$

where we set  $k_{r+1} = \cdots = k_{D+2} := 0$  and  $M := \text{lcm}\{2+k_i, i = 1, \dots, D+2\}$  [28]. This claim has been considerably substantiated in [33], though here we will not go into details of its precise meaning in the presence of quantum corrections. For small  $D$ ,  $D \in \{1, 2\}$ , however, quantum corrections are not expected in the description of the relevant moduli spaces, so that this claim can be made much more precise. Indeed, the Definitions 1.1 and 2.3 together with the properties of Gepner models discussed in Appendix C ensure that these models are associated to elliptic curves if  $D = 1$  or a real four-torus or  $K3$  surface if  $D = 2$ . Specifically for the Gepner model  $(2)^2$  we hence expect a geometric interpretation on the elliptic curve

$$y_0^2 = y_1^4 + y_2^4 \quad \text{in } \mathbb{CP}_{2,1,1},$$

i.e. on an elliptic curve with modulus  $\tau = i$  by Appendix A. In fact,  $(2)^2$  agrees with the toroidal SCFT at central charges  $c = \bar{c} = 3$  which is specified by the two moduli  $\tau = \rho = i$ . This claim is well established in the literature [78]. However, since I will need the explicit identifications of fields in these two theories, let me sketch the proof.\*

We wish to identify two  $N = (2, 2)$  SCFTs at central charges  $c = \bar{c} = 3$ , both of which are invariant under spectral flow and contain only fields with integral  $\mathfrak{u}(1)$  charges in their NS sectors. Moreover, one checks that for  $\tau = \rho = i$  the partition function of the toroidal theory, which can be obtained from (1.4), agrees with the one constructed for  $(2)^2$  in (C.4). It remains to be shown that  $(2)^2$  decomposes into the tensor product of the fermionic theory at  $c = \bar{c} = 1$  describing a Dirac fermion, and a bosonic theory at  $c = \bar{c} = 2$  with two further  $\mathfrak{u}(1)$  currents on each side, such that the relevant charge lattice is  $\Gamma_{i,i}$  as given in (1.5). To this end, one starts by using (B.3) to determine all fermionic holomorphic fields of  $(2)^2$  with conformal weights  $(\frac{1}{2}, 0)$ . There are only two such linearly independent fields, realized by the operators of two-fold left-handed spectral flows. Hence taking  $\mathfrak{u}(1)$  charges into account we readily identify

$$\psi_+ = \Phi_{-2,2;0,0}^0 \otimes \Phi_{-2,2;0,0}^0, \quad \psi_- = \Phi_{2,2;0,0}^0 \otimes \Phi_{2,2;0,0}^0. \quad (\text{D.1})$$

Recall that these fields are simple currents, and by (C.2) they indeed realize the OPE of a Dirac fermion. Moreover, since the analogous simple currents exist on the right hand side,  $(2)^2$  splits into a tensor product of a bosonic theory  $\mathcal{B}$  at central charges  $c = \bar{c} = 2$  with the

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\*The proof I gave together with W. Nahm in [30, Theorem 3.2] unfortunately contains typos and a gap, which I also wish to correct here. As we shall see, these mistakes do not influence any other results in that publication.

fermionic theory which describes the Dirac fermion. The superpartners of the  $\psi_{\pm}$  give two further Hermitean conjugate  $\mathfrak{u}(1)$  currents on each side of the theory  $\mathcal{B}$ ,

$$j_{\pm} = \Phi_{\mp 2,0;0,0}^0 \otimes \Phi_{\mp 2,2;0,0}^0 - \Phi_{\mp 2,2;0,0}^0 \otimes \Phi_{\mp 2,0;0,0}^0, \quad (\text{D.2})$$

such that the real and imaginary parts of  $\sqrt{2}j_{\pm}$  obey (1.2), and similarly on the right-hand side. Hence this theory indeed is a bosonic toroidal CFT, and it remains to determine its charge lattice  $\Gamma_{\tau,\rho}$ . The primary fields of  $\mathcal{B}$  are obtained as follows: Each orbit under the action of  $\psi_+$  and its right handed analog on the NS sector of  $(2)^2$  contributes one such primary field, namely the one with lowest conformal weights in the orbit. Using (B.3) we therefore find that the following fields contribute as left or right hand components of primary fields in  $\mathcal{B}$ , with notations as in (B.1):

$$\begin{aligned} & \psi_{0,0}^0 \otimes \psi_{0,0}^0, \quad \psi_{4,2}^0 \otimes \psi_{0,0}^0, \quad \psi_{0,0}^0 \otimes \psi_{4,2}^0, \quad \psi_{4,2}^0 \otimes \psi_{4,2}^0, \\ & \psi_{2,0}^0 \otimes \psi_{2,2}^0, \quad \psi_{2,2}^0 \otimes \psi_{2,0}^0, \quad \psi_{-2,0}^0 \otimes \psi_{-2,2}^0, \quad \psi_{-2,2}^0 \otimes \psi_{-2,0}^0, \\ & \psi_{1,0}^1 \otimes \psi_{-1,0}^1, \quad \psi_{-1,0}^1 \otimes \psi_{1,0}^1, \quad \psi_{3,2}^1 \otimes \psi_{-3,2}^1, \quad \psi_{-3,2}^1 \otimes \psi_{3,2}^1, \\ & \psi_{1,0}^1 \otimes \psi_{3,2}^1, \quad \psi_{-1,0}^1 \otimes \psi_{-3,2}^1, \quad \psi_{3,2}^1 \otimes \psi_{1,0}^1, \quad \psi_{-3,2}^1 \otimes \psi_{-1,0}^1. \end{aligned}$$

Let us denote the  $\mathfrak{u}(1)$  currents of the two minimal model factors of  $(2)^2$  by  $j = \frac{i}{\sqrt{2}}\partial\varphi$ ,  $j' = \frac{i}{\sqrt{2}}\partial\varphi'$ , respectively, such that  $j + j' = J = i:\psi_-\psi_+:$ . Then apart from the  $\mathfrak{u}(1)^2 \oplus \mathfrak{u}(1)^2$  current algebra generated by  $j_{\pm}, \bar{j}_{\pm}$  the theory  $\mathcal{B}$  allows a further four-dimensional space of holomorphic primaries at weight  $(1,0)$ , generated by

$$\begin{aligned} & j - j', \quad \Phi_{4,2;0,0}^0 \otimes \Phi_{4,2;0,0}^0, \\ & \Phi_{2,0;0,0}^0 \otimes \Phi_{2,2;0,0}^0 + \Phi_{2,2;0,0}^0 \otimes \Phi_{2,0;0,0}^0, \quad \Phi_{-2,0;0,0}^0 \otimes \Phi_{-2,2;0,0}^0 + \Phi_{-2,2;0,0}^0 \otimes \Phi_{-2,0;0,0}^0, \end{aligned}$$

and similarly for antiholomorphic fields of weights  $(0,1)$ . Hence without loss of generality we can assume that the lattice  $\Lambda$  generated by  $\lambda_1, \lambda_2$  in (1.5) contains vectors  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $e_2 = \begin{pmatrix} a \\ b \end{pmatrix}$  with  $a^2 + b^2 = 1$ .

On the other hand, using (B.1) one finds that the eight primaries

$$\begin{aligned} & \Phi_{1,0;1,0}^1 \otimes \Phi_{-1,0;-1,0}^1, \quad \Phi_{-1,0;-1,0}^1 \otimes \Phi_{1,0;1,0}^1, \quad \Phi_{1,0;-3,2}^1 \otimes \Phi_{-1,0;3,2}^1, \quad \Phi_{-1,0;3,2}^1 \otimes \Phi_{1,0;-3,2}^1, \\ & \Phi_{1,0;-1,0}^1 \otimes \Phi_{-1,0;-3,2}^1, \quad \Phi_{-1,0;1,0}^1 \otimes \Phi_{1,0;3,2}^1, \quad \Phi_{1,0;3,2}^1 \otimes \Phi_{-1,0;1,0}^1, \quad \Phi_{-1,0;-3,2}^1 \otimes \Phi_{1,0;-1,0}^1 \end{aligned}$$

in  $\mathcal{B}$  together with the Virasoro fields generate all fields in  $\mathcal{B}$  by means of the OPE. All the eight primaries in the above list have the minimal non-zero conformal weights which occur in  $\mathcal{B}$ ,  $h = \bar{h} = \frac{1}{4}$ . Denoting by  $\Lambda^*$  the lattice generated by  $\lambda_1^*, \lambda_2^*$  in (1.5), this means that the lattice  $\Gamma_{\tau,\rho}$  is generated by four vectors of the form

$$\frac{1}{\sqrt{2}}(\lambda - B\lambda, -\lambda - B\lambda), \quad \frac{1}{\sqrt{2}}(\tilde{\lambda} - B\tilde{\lambda}, -\tilde{\lambda} - B\tilde{\lambda}), \quad \frac{1}{\sqrt{2}}(\lambda^*, \lambda^*), \quad \frac{1}{\sqrt{2}}(\tilde{\lambda}^*, \tilde{\lambda}^*), \quad \lambda, \tilde{\lambda} \in \Lambda, \quad \lambda^*, \tilde{\lambda}^* \in \Lambda^*,$$

where  $\lambda^*$  and  $\tilde{\lambda}^*$  are generators of the lattice  $\Lambda^*$  with length 1. Together with  $e_1, e_2 \in \Lambda$ , and since  $\Lambda^*$  is indeed the dual lattice of  $\Lambda$  when we identify  $\mathbb{R}^2 \cong (\mathbb{R}^2)^*$  by means of the standard Euclidean scalar product, this implies that without loss of generality  $\lambda^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\tilde{\lambda}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\lambda^* = e_1, \tilde{\lambda}^* = e_2$ . Hence  $\Lambda = \Lambda^* = \mathbb{Z}^2$ , meaning  $\tau = \rho = i$  as claimed.

## Appendix D.2 The Gepner models $(2)^4$ and $(\hat{2})^4$

By the above the fermionic tensor product  $\mathcal{T}_{1,0,0,1} := (2)^2 \otimes (2)^2$  of two Gepner models  $(2)^2$  is the toroidal SCFT on a complex two-torus  $A_{1,0,1} = \mathbb{C}^2/\sim$  which is the product of



two elliptic curves with moduli  $\tau = \rho = i$  each. On  $A_{1,0,1}$  and with respect to standard coordinates  $(z_1, z_2)$  of  $\mathbb{C}^2$  we have  $z_k \sim z_k + 1 \sim z_k + i$ , and the theory  $\mathcal{T}_{1,0,0,1}$  has vanishing B-field. In the following we denote the left handed Dirac fermions of the two tensor factors  $(2)^2$  of  $\mathcal{T}_{1,0,0,1}$  by  $\psi_\pm^1, \psi_\pm^2$ , respectively, and their superpartners by  $j_\pm^1, j_\pm^2$ . The theory  $\mathcal{T}_{1,0,0,1}$  enjoys a natural symmetry of order 4, which is induced by the geometric symmetry  $(z_1, z_2) \mapsto (iz_1, -iz_2)$  of  $A_{1,0,1}$ , or more precisely by

$$(\psi_\pm^1, \psi_\pm^2) \mapsto (\pm i \psi_\pm^1, \mp i \psi_\pm^2), \quad (j_\pm^1, j_\pm^2) \mapsto (\pm i j_\pm^1, \mp i j_\pm^2).$$

Given the identifications (D.1) and (D.2) this means that in Gepner language on  $(2)^2 \otimes (2)^2$  we are using the symmetry

$$\iota: \bigotimes_{j=1}^4 \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j} \mapsto e^{\frac{2\pi i}{8}[(\overline{m}_1 - m_1) - (\overline{m}_3 - m_3)]} \bigotimes_{j=1}^4 \Phi_{m_j, s_j; \overline{m}_j, \overline{s}_j}^{l_j}. \quad (\text{D.3})$$

Now recall our description of Gepner models as orbifolds in terms of the GSO projection by  $\zeta_M$  in (C.1), where in our case  $M = 4$ . A field  $\Phi^{(1)} \otimes \Phi^{(2)}$  of  $(2)^2 \otimes (2)^2$ , with  $\Phi^{(l)}$  belonging to the sector of the  $l^{\text{th}}$  tensor factor  $(2)^2$  twisted by  $\zeta_4^{b_l}$ , is invariant under the above symmetry iff  $b_1 = b_2$ . Again by our description of Gepner models this implies directly that the above orbifold, which was induced by the standard geometric  $\mathbb{Z}_4$ -symmetry of  $\mathcal{T}_{1,0,0,1}$ , gives the Gepner model  $(2)^4$ . This was already shown in [30, Theorem 3.5]. Moreover, we also obtain directly the result [30, Theorem 3.3] that the standard geometric  $\mathbb{Z}_2$ -orbifold of  $\mathcal{T}_{1,0,0,1}$ , i.e. the orbifold by  $\iota^2$  above, yields the model  $(\hat{2})^4$  which is obtained from  $(2)^4$  by means of the  $\mathbb{Z}_2$ -orbifold by  $[2, 2, 0, 0]$  with notations as in (C.3). In [30], the proof that  $(\hat{2})^4$  agrees with the standard  $\mathbb{Z}_2$ -orbifold of  $\mathcal{T}_{1,0,0,1}$  was given independently of the identification of  $(2)^2$ . Note that the above corrected field identification for  $(2)^2$  now directly induces the precise identification [30, (3.8)] for  $(\hat{2})^4$ .

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